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Off-Diagonal Elements of Normal Matrices*

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Let A be an $n \times n$ complex matrix, and let $W(A)$ denote the numerical range of A. In this paper, results of Parker and Mirsky are shown to be a consequence of the following fact. If A is normal, then the maximum value of $|(Ax,y)|$ as x and y run over all ortho-normal pairs in C^n coincides with the radius of the smallest closed disk in C which contains $W(A)$.

Key words: Constrained extrema of quadratic forms; numerical range.

1. Introduction

For an $n \times n$ complex matrix, Parker [1] [1] has proven that the maximum value, for fixed A, of

$$|(Ax, y)| \tag{1}$$

subject to the constraint $||x|| = ||y|| = 1$ is the maximum singular value, or *Hilbert norm*, $s_1(A)$ of A. In fact, it is clear that the maximum value in (1) occurrs exactly when x is a unit eigenvector in C^n corresponding to the maximum eigenvalue of A^*A, and $y = Ax/||Ax||$.

If in (1) we require that $x = y$, then the maximum value of (1) is the *numerical radius* of A, $r(A)$. A moments reflection yields that

$$r(A) = \max_{U \text{ unitary}} |(U^*AU)_{1,1}|. \tag{2}$$

Relations between $s_1(A)$ and $r(A)$ have been studied. See for example [2,3].

Parker in [1] also investigated the maximum value of (1) when A is Hermitian and x,y range over all orthonormal pairs. In this case, the maximum value of (1) is $1/2(\lambda_1(A) - \lambda_n(A))$, where $\lambda_1(A) \geqq \ldots \geqq \lambda_n(A)$ are the eigenvalues of A. Mirsky later obtained [4] that if A is normal with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\max_{x,\, y \text{ orthonormal}} |(Ax, y)| \leqq \max_{i,\, j} \left| \frac{\lambda_i - \lambda_j}{\sqrt{3}} \right|. \tag{3}$$

We remark that for any A, the left-hand side of (3) is identical with

$$\max_{U \text{ unitary}} |(U^*AU)_{1,2}|. \tag{4}$$

In this paper we completely characterize this maximum value for A, and obtain the results of Parker and Mirsky as corollaries.

2. Notation

(i) A will denote an $n \times n$ complex matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$.

(ii) $W(A) = \{(Ax,x) \mid ||x|| = 1\}$ is the *numercial range* of A. It is well known that if A is normal, then $W(A) = \mathcal{H}(\lambda_1, \ldots, \lambda_1)$, the convex hull of $\lambda_1, \ldots, \lambda_n$.

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1 Figures in brackets indicate the literature references at the end of this paper.

(iii) $S(A)$ will denote $\{(Ax,y)|x,y \text{ orthonormal}\}$. A theorem of Marcus and Robinson [5] states that $S(A)$ is a closed disk centered at 0 in the complex plane. We will use the notation $\rho(A)$ for the radus of $S(A)$.

The following lemma is an immediate consequence of definitions (ii) and (iii).

LEMMA: *Let* $\alpha, \beta \in C$. *Then*

(i) $S(\alpha A+\beta I_n)=|\alpha|S(A)$

(ii) $W(\alpha A+\beta I_n)=\alpha W(A)+\beta$.

We are now in a position to state the following:
THEOREM: *If A is normal, then* $\rho(A)$ *is the radius of the smallest closed disc containing* $W(A)$.

PROOF: By the lemma, there is no loss of generality in assuming that the smallest closed disk containing $W(A)$ is the unit circle. Since A is normal this implies that the Hilbert norm of A is 1, and by Cauchy-Schwarz we have

$$\max_{x,y \text{ orthonormal}} |(Ax,y)| \leqq \max_{||x||=1} ||Ax||=1.$$

Hence $\rho(A)\leqq 1$, and to complete the proof we need only exhibit an orthonormal pair x,y for which $|(Ax,y)|=1$. Since $W(A)$ is closed and convex, and the unit disk is the (unique) smallest disk containing $W(A)$, we must have at least one of the following holding:

(i) There exist extreme points λ_1, λ_2 of $W(A)$ such that $\overline{\lambda_1\lambda_2}$ is a diameter of the unit disk.
(ii) There exist extreme points $\lambda_1, \lambda_2, \lambda_3$ of $W(A)$ such that the triangle $\Delta(\lambda_1, \lambda_2, \lambda_3)$ is an acute triangle inscribed in the unit circle.

Since A is normal, these extreme points of $W(A)$ are eigenvalues of A. If (i) holds, then we have eigenvalues $\lambda_1, -\lambda_1$ of A with $|\lambda_1|=1$. Since A is normal there exists an orthonormal pair of eigenvectors u,v corresponding to $\lambda_1, -\lambda_1$ respectively. The pair

$$x=\frac{u+v}{\sqrt{2}}, \qquad y=\frac{u-v}{\sqrt{2}}$$

is orthonormal, and we calculate that

$$|(Ax, y)|=\frac{1}{2}|\lambda_1-(-\lambda_1)|=1,$$

and the proof is finished in this case.

If case (ii) holds, let u, v, w be a set of orthonormal eigenvectors corresponding to $\lambda_1, \lambda_2, \lambda_3$ respectively. Also, let a, b, c be the unique positive numbers which satisfy

$$a+b+c=1$$

$$a\lambda_1+b\lambda_2+c\lambda_3=0.$$

Then, let

$$x=\sqrt{a}\,u+\sqrt{b}\,v+\sqrt{c}\,w.$$

$$y=\sqrt{a}\,\lambda_1 u+\sqrt{b}\,\lambda_2 v+\sqrt{c}\,\lambda_3 w.$$

Again, trivial calculations yield that x, y is an orthonormal pair, and that $(Ax, y)=1$.

COROLLARY 1: *If* $W(A)$ *is a line segment with endpoints* λ_1, λ_n, *then* $\rho(A)=\frac{1}{2}|\lambda_1-\lambda_n|$.

PROOF: If $W(A)$ is a line segment, then A is essentially Hermitian and we may apply the theorem.

COROLLARY 2: *Let A be a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then*

$$\frac{\sqrt{3}}{2}\rho(A) \leqq \max_{i,j} \frac{|\lambda_i - \lambda_j|}{2} \leqq \rho(A).$$

PROOF: This follows from the theorem and the geometric fact that if a non-obtuse triangle with longest side s is inscribed in a circle of radius 1, then $\sqrt{3} \leqq s \leqq 2$.

In light of the theorem it is of interest to be able to compute the radius of the smallest circle containing a triangle Δ with sides of length a, b, and c. For non-obtuse triangles we may use the following proposition. The proof, which is omitted, uses the formula for $\cos(\alpha+\beta)$ and the law of cosines.

PROPOSITION 1: Let the triangle Δ have sides of length a, b, c. Then the radius of the circle circumscribed about Δ is

$$\frac{abc}{\sqrt{2E_2(a^2,b^2,c^2)-(a^4+b^4+c^4)}},$$

where E_2 denotes the second elementary symmetric function.

The theorem may be restated as follows: "For A normal,

$$\rho(A) = \min_{\alpha} ||A-\alpha I_n|| = \min_{\alpha} \max_{||x||=1} ||Ax-\alpha x||."$$

The question of whether this result could hold in general is put in a somewhat different perspective by the following proposition.

PROPOSITION 2: For any A,

$$\rho(A) = \max_{||x||=1} \min_{\alpha} ||Ax-\alpha x||\cdot$$

PROOF:

$$\begin{aligned}\rho(A) &= \max_{x,y \text{ o.n.}} |(Ax,y)| \\ &= \max_{||x||=1} \max_{\substack{||y||=1 \\ (x,y)=0}} |(Ax,y)| \\ &= \max_{||x||=1} ||Ax-(Ax,x)x|| \\ &= \max_{||x||=1} \min_{\alpha} ||Ax-\alpha x||.\end{aligned}$$

Certainly we have

$$\max_{||x||=1} \min_{\alpha} ||Ax-\alpha x|| \leqq \min_{\alpha} \max_{||x||=1} ||Ax-\alpha x||,$$

and the question of whether equality holds remains open.

Finally it shouid be noted that the theorem is false in general, even for convexoid A (A is *convexoid* if $W(A)=\mathcal{H}(\lambda_1, \ldots, \lambda_n)$), as the following matrix shows. Let

$$A=\begin{bmatrix} 1+i & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-i & 0 & 0 & 0 & 0 \\ 0 & 0 & -1+i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the radius of the smallest circle containing $W(A)$ is $\sqrt{2}$, while $\rho(A)=||A||=2$.

3. References

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