JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 81B, Nos. 1 and 2, January–June 1977

Electromagnetism in Non-Riemannian Space

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(March 31, 1977)

Maxwell's equations can be interpreted as two conservation laws in a four-dimensional geometric manifold, expressed as the vanishing of a divergence and of a curl. These natural derivatives are invariant under holonomic coordinate transformations in any geometric manifold, and contain no reference to properties of the manifold such as its metric tensor and linear connection.

The relation between the *D–H* and *E–B* fields is classically determined by the metric tensor. If a general asymmetric connection is considered, the field relations can still be derived from Hamilton's principle with the addition of an anholonomic constraint.

The basic effect of the inclusion of asymmetry (a non-vanishing torsion) is to destroy the parallelism between the Poynting vector $\mathbf{E}\times\mathbf{H}$ and the momentum vector $\mathbf{D}\times\mathbf{B}$.

Key words: Anholonomic; constraint; electromagnetism; Hamilton; Maxwell; momentum; Poynting; variational.

Maxwell's equations can be interpreted as conservation laws:

$$\partial_\lambda \mathcal{J}^\lambda=0 \tag{1}$$

$$\partial_{[\lambda} F_{\mu\nu]}=0. \tag{2}$$

These equations are invariant under holonomic coordinate transformations in any geometric manifold.

In a holonomic coordinate system, (1) and (2) imply

$$\mathcal{J}^\lambda=\partial_\mu \mathcal{G}^{[\lambda\mu]} \tag{3}$$

$$F_{\mu\nu}=2\partial_{[\mu} A_{\nu]} \tag{4}$$

and, in fact, (1) and (2) become identities if $\mathcal{J}$ and F are defined by (3), (4), with $\mathcal{G}^{\lambda\mu}$ and A_ν arbitrary.

Equations (3) and (4) can be combined to yield the identity

$$\partial_\pi(F_{\mu\lambda}\mathcal{G}^{\lambda\pi})+\frac{1}{2}\mathcal{G}^{\lambda\pi}\partial_\mu F_{\lambda\pi}\equiv F_{\mu\lambda}\mathcal{J}^\lambda \tag{5}$$

which has a physical interpretation in terms of energy and stress.

Consider an arbitrary scalar density function $\mathcal{L}$ of A_ν, $\partial_{[\mu}A_{\nu]}$, and various parameters of the manifold. Define

$$F_{\mu\nu}\overset{\mathrm{d}}{=}2\partial_{[\mu}A_{\nu]} \tag{6}$$

$$\mathcal{G}^{\mu\nu}\overset{\mathrm{d}}{=}\frac{\partial\mathcal{L}}{\partial(\partial_{[\mu}A_{\nu]})} \tag{7}$$

$$\mathcal{J}^\lambda\overset{\mathrm{d}}{=}\partial_\mu\mathcal{G}^{\lambda\mu} \tag{8}$$

Expressing the Lie derivative of $\mathcal{L}$ in two ways [1] [1] yields (5) as an identity arising from the condition that the arguments of $\mathcal{L}$ must combine to yield a scalar density. Thus any scalar density function of an arbitrary A_ν and its rotation yields (5) as a *mathematical* identity among the quantities defined by (6), (7), and (8).

[1] Figures in brackets indicate the literature references at the end of this paper.

To attach physical meaning to all these identities, we assume a 4-current $\mathcal{J}^{\lambda}$, satisfying (1), as a *source*, and require that the resulting fields satisfy Hamilton's variational principle. Since (1) and (2) are valid in any geometric manifold, $\mathcal{J}^{\lambda}$ and A_{ν} are gauge invariant, i.e., if we assign a metric tensor $g^{\alpha\beta}$ to the manifold, $\mathcal{J}^{\lambda}$ and A_{ν} are invariant under conformal transformations of $g^{\alpha\beta}$. The gauge invariance of $F_{\mu\nu}$, $\mathcal{G}^{\mu\nu}$, and $\mathcal{L}$ follows.

Since the only gauge-invariant tensor density associated with the manifold is $g^{1/2}g^{\alpha\rho}g^{\beta\sigma}$, a gauge-invariant $\mathcal{L}$ comprises terms of the form

$$\mathcal{J}^{\lambda}A_{\lambda}$$

$$g^{1/2}g^{\alpha\rho}g^{\beta\sigma}\partial_{\alpha}A_{\beta}\partial_{[\rho}A_{\sigma]}$$

$$g^{1/2}g^{\alpha\rho}g^{\beta\sigma}S_{\alpha\beta}{}^{\lambda}A_{\lambda}\partial_{[\rho}A_{\sigma]}$$

where $S_{\alpha\beta}{}^{\lambda}$ is any gauge-invariant tensor.

If the manifold is endowed with a linear connection, its torsion satisfies the conditions on $S_{\alpha\beta}{}^{\lambda}$. A linear connection has no other gauge invariant parameters of this form, so it is natural to consider $S_{\alpha\beta}{}^{\lambda}$ to be the torsion of a connection and investigate its effects on $\mathcal{L}$ as a Lagrangian.

If we add a torsion term to the familar Lagrangian for a riemann space, we have

$$\mathcal{L}=-\mathcal{J}^{\lambda}A_{\lambda}+g^{1/2}g^{\alpha\rho}g^{\beta\sigma}\partial_{[\alpha}A_{\beta]}\partial_{[\rho}A_{\sigma]}+Cg^{1/2}g^{\alpha\rho}g^{\beta\sigma}S_{\alpha\beta}{}^{\lambda}A_{\lambda}\partial_{[\rho}A_{\sigma]} \tag{9}$$

which gives

$$\mathcal{G}^{\alpha\beta}=g^{1/2}F^{\alpha\beta}+Cg^{1/2}S^{\alpha\beta\lambda}A_{\lambda} \tag{10}$$

so that $\mathcal{L}$ can also be written as

$$\mathcal{L}=-\mathcal{J}^{\lambda}A_{\lambda}+\frac{1}{2}\mathcal{G}^{\alpha\beta}F_{\alpha\beta}-\frac{1}{4}g^{1/2}F^{\alpha\beta}F_{\alpha\beta}. \tag{11}$$

Now $\mathcal{L}$ satisfies Hamilton's principle if the allowed variations of A_{λ} are subjected to the anholonomic constraint [2]

$$S^{\alpha\beta\lambda}F_{\alpha\beta}\delta A_{\lambda}=0. \tag{12}$$

If the constant C is taken to be -2, then

$$\mathcal{G}^{\alpha\beta}=2g^{1/2}g^{\alpha\rho}g^{\beta\sigma}\nabla_{[\rho}A_{\sigma]} \tag{13}$$

which is a covariant generalization of the riemannian case

$$\mathcal{G}^{\alpha\beta}=2g^{1/2}g^{\alpha\rho}g^{\beta\sigma}\partial_{[\rho}A_{\sigma]}. \tag{14}$$

Now consider the effect of the constraint on the change of fields resulting from a change of the 4-current source, $\mathcal{J}^{\lambda}$:

$$d\mathcal{L}=-d(\mathcal{J}^{\lambda}A_{\lambda})+\frac{1}{2}g^{1/2}F^{\alpha\beta}dF_{\alpha\beta}+\frac{C}{2}g^{1/2}S^{\alpha\beta\mu}A_{\mu}dF_{\alpha\beta}+\frac{C}{2}g^{1/2}S^{\alpha\beta\mu}F_{\alpha\beta}dA_{\mu}. \tag{15}$$

The last term vanishes by virtue of the constraint, leaving

$$d\mathcal{L}=-d(\mathcal{J}^{\lambda}A_{\lambda})+\frac{1}{2}\mathcal{G}^{\alpha\beta}dF_{\alpha\beta}. \tag{16}$$

Allowing $\mathcal{J}^{\lambda}$ to build up from zero to its final state gives

$$\mathcal{L} = -\mathcal{J}^{\lambda}A_{\lambda}+\frac{1}{2}\int \mathfrak{G}^{\alpha\beta}dF_{\alpha\beta} \tag{17}$$

which can be expressed in terms of 3-dimensional fields as

$$\mathcal{L} = -J\cdot A+\rho\phi+\int H\cdot dB-\int D\cdot dE. \tag{18}$$

Note that the torsion does not explicitly appear in (16)–(18); its effect is implicit in the fields resulting from a given source.

On the other hand, if we compute the Poynting vector, $\mathbf{E}\times\mathbf{H}$, and the momentum, $\mathbf{D}\times\mathbf{B}$, from (10), we find interesting differences. The Poynting vector contains only spatial asymmetry terms, $S^{ij\lambda}$; $i,j=1, 2, 3,$. The momentum contains only $S^{0j\lambda}$, representing time-space asymmetries. Thus, in general, the Poynting and momentum vectors will not be parallel.

References

[1] Schouten, J. A., Ricci-Calculus, 2nd edition (1954) (Springer-Verlag, Berlin), p. 115.

[2] See, for example, Whittaker, E. T., Analytical Dynamics, 4th edition (1944) (Dover Publication, New York), p. 33 ff.

(Paper 81B 1 & 2–458)