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A Note on Pseudointersection Graphs*

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Pseudointersection graphs are defined and a parameter called the pseudointersection number of a graph, denoted $\omega^*(G)$ and closely related to the intersection number of G, denoted $\omega(G)$, is introduced. Relations between these parameters and conditions for them to be equal are examined. The problem of computing $\omega^*(G)$ is examined.

Key words: Clique; clique graph; intersection graph; pseudointersection graph; set covering.

1. Introduction

A graph $G = (V,E)$ where V and E are the vertex and edge sets shall be considered to be a simple graph (i.e., finite, undirected and without loops or multiple edges), and all terms used shall be consistent with their definitions in [3][1].

If S is a set and $F = \{S_1,S_2, \ldots, S_p\}$ is a family of distinct nonempty subsets of S whose union is S, then the *intersection graph* of F, denoted by $\Omega(F)$, is the graph with $V\Big(\Omega(F)\Big) = F$ such that S_i and S_j are adjacent if and only if (iff) $i \neq j$ and $S_i \cap S_j \neq \emptyset$. A graph G is an intersection graph on S if there exists such a family F for which $G \approx \Omega(F)$. Every graph G is an intersection graph on some finite set [7], and the *intersection number* $\omega(G)$ is the minimum number of elements in a set S such that G is an intersection graph on S.

If $|S| = n$ then, as defined by S. Hedetniemi [5], a *representation* of G as an intersection graph on S is a one to one function, $r{:}V(G) \rightarrow \{0,1\}^n$, such that for $u,v \in V(G)$ one has $(u,v) \in E(G)$ iff $r(u)$ and $r(v)$ have a 1 in a common coordinate position, and if $1 \leq i \leq n$ then there is some $v \in V(G)$ such that $r(v)$ has a 1 in the ith coordinate position.

For the complete graph K_3 on vertices v_1, v_2, and v_3 we have $\omega(K_3) = 3$. If $S = \{a,b,c\}$ then one can choose, for example, $S_1 = \{a\}$, $S_2 = \{a,b\}$, and $S_3 = \{a,c\}$ or $S_1 = \{a,b\}$, $S_2 = \{b,c\}$ and $S_3 = \{a,c\}$. In the former case it is clear that elements b and c are needed only to make the S_i's distinct and do nothing to indicate adjacency. Equivalently, for $r{:}V(K_3) \rightarrow \{0,1\}^3$ with $r(v_1) = (1,0,0)$, $r(v_2) = (1,1,0)$ and $r(v_3) = (1,0,1)$, only the first coordinate has more than one 1 in it. As another example, the graph $K_4 - x$ is given in figure 1 as an intersection graph, and, in this case, element c of S is not necessary to indicate the adjacency of any two vertices. The size required for S can be reduced by eliminating these "fillers" used only to obtain distinct representations of each vertex.

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[1] Figures in brackets indicate the literature references at the end of this paper.

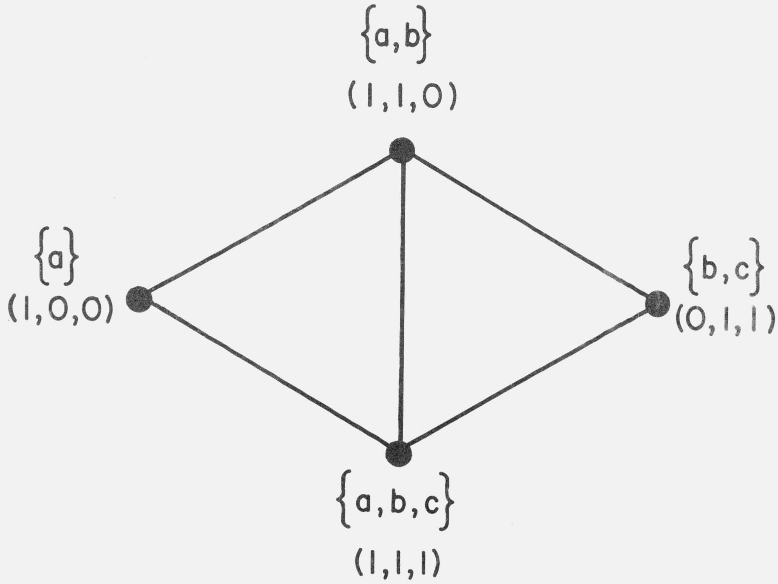

FIGURE 1. *Graph* $K_4 - x$ *as an intersection graph.*

If S is a set and $F = \{S_1, S_2, \ldots S_p\}$ is a family of subsets of S (not necessarily nonempty or distinct) whose union is S, then the *pseudointersection graph of F*, denoted by $\Omega^*(F)$, is the graph with $V\left(\Omega^*(F)\right) = F$ such that S_i and S_j are adjacent iff $i \neq j$ and $S_i \cap S_j \neq \emptyset$. Note that $S_i = \emptyset$ implies S_i corresponds to an isolated vertex. A graph G is a pseudointersection graph on S if there exists such a family F for which $G \approx \Omega^*(F)$. The *pseudointersection number of G*, denoted $\omega^*(G)$, is the minimum number of elements in a set S such that G is a pseudointersection graph on S. In particular, $\omega^*(K_t) = 0$, and $t \geq 2$ implies $\omega^*(K_t) = 1$.

If $|S| = n$ then a pseudorepresentation of G as an intersection graph on S is a function, $r: V(G) \to \{0,1\}^n$, such that for $u,v \in V(G)$ one has $(u,v) \in E(G)$ iff $r(u)$ and $r(v)$ have a 1 in a common coordinate position. The requirements that r be one to one and that $1 \leq i \leq n$ implies some $v \in V(G)$ has a 1 in the ith coordinate position have been dropped. However, if the ith component of $r(v)$ is 0 for every $v \in V(G)$, then clearly $\omega^*(G) \leq n - 1$.

2. Computing $\omega^*(G)$

Since every representation is a pseudorepresentation one obtains the following.

PROPOSITION 1: *For any graph* G, $\omega^*(G) \leq \omega(G)$.

For a graph G, $\theta(G)$ has been used to denote the minimum number of complete subgraphs of G which contain all the vertices of G. If one lets $\theta'(G)$ denote the minimum number of vertex disjoint complete subgraphs of G which contain all the vertices of G, then it is easy to see that $\theta(G) = \theta'(G)$. Now let $\theta_1(G)$ be the minimum number of complete subgraphs of G which contain all the edges of G, and let $\theta_1'(G)$ be the minimum number of edge disjoint complete subgraphs of G which contain all the edges of G. For example, $\theta(K_4 - x) = \theta_1(K_4 - x) = 2$ and $\theta_1'(K_4 - x) = 3$. Clearly $\theta_1(G) \leq \theta_1'(G)$ for every graph G. Note that $\omega^*(K_t) = 0 = \theta_1(K_t)$.

THEOREM 2: *For any graph* G, $\omega^*(G) = \theta_1(G)$.

PROOF: Suppose $\omega^*(G) = k > 0$, and let $r: V(G) \to \{0,1\}^k$ be a pseudorepresentation of G. Let S_i be the set of all vertices v in $V(G)$ for which $r(v)$ has a 1 in the ith coordinate $(1 \leq i \leq k)$. Now the subgraph generated by S_i, denoted $\langle S_i \rangle$, is complete since $u,v \in S_i$ implies u and v have a 1 in a common coordinate position. If $(u,v) \in E(G)$ then u and v have a 1 in some common coordinate, say the ith. Hence $(u,v) \in \langle S_i \rangle$. Thus $\langle S_1 \rangle, \langle S_2 \rangle, \ldots, \langle S_k \rangle$ are complete subgraphs containing every edge of G, and $\theta_1(G) \leq \omega^*(G)$.

Suppose $\theta_1(G) = k$, and let $S_1, S_2, \ldots, S_k$ be the point sets of complete subgraphs such that every edge of G is in some $\langle S_i \rangle$. Define $r: V(G) \to \{0,1\}^k$ by $r(v) = (e_1, e_2, \ldots, e_k)$ where $e_1 = 1$ if $v \in S_i$ and $e_i = 0$ if $v \in S_i$. It is easy to see that r is a pseudorepresentation of G. Thus $\omega^*(G) \leq \theta_1(G)$.

While $\theta_1 = \omega^*$, θ_1' and ω are independent parameters. For example, $\omega^*(K_t) = \theta_1(K_t) = \theta_1'(K_t) = 1 < \omega(K_t)$ when $t \geq 2$. For the graph G of figure 2, $\theta_1(G) = \omega^*(G) = \omega(G) = 3$ and $\theta_1'(G) = 4$.

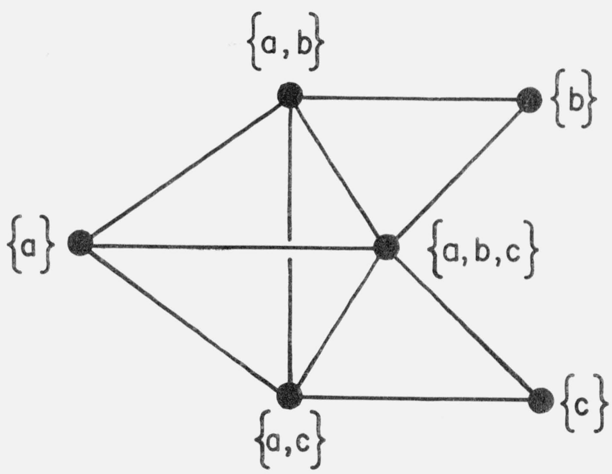

FIGURE 2. *A graph with* $\omega(G) < \theta_1'(G)$.

If $v \in V(G)$ then the *neighborhood of* v, denoted $N(v)$, is the set of vertices adjacent to v, and the *closed neighborhood of* v, denoted $N[v]$, is $N(v) \bigcup \{v\}$.

THEOREM 3: *If* G *has no isolated vertices and for any two distinct vertices,* u *and* v, $N[u] \neq N[v]$, *then* $\omega^*(G) = \omega(G)$.

PROOF: Let $r: V(G) \to \{0,1\}^n$ be a pseudorepresentation of G where $\omega^*(G) = n$. Suppose $r(v_1) = r(v_2)$. Since G has no isolated vertices, $r(v_1) \neq (0,0,\ldots,0)$. Hence $r(v_1)$ and $r(v_2)$ have a 1 in a common coordinate and so v_1 and v_2 are adjacent. Now u is adjacent to v_1 iff $r(u)$ and $r(v_1) = r(v_2)$ have a 1 in a common coordinate iff u is adjacent to v_2. Thus $N[v_1] = N[v_2]$.

This contradiction implies that r is a one to one function. As already noted, there cannot be an i with $1 \leq i \leq n$ such that $r(v)$ is 0 in the ith component for every $v \in V(G)$. That is, r is actually a representation. This implies $\omega(G) \leq \omega^*(G)$. Consequently $\omega(G) = \omega^*(G)$.

COROLLARY 3.1.1: *If* G *is triangleless and each component has at least three vertices, then* $\omega(G) = \omega^*(G)$.

The following is an easy consequence of the fact that $\omega^*(G) = \theta_1(G) \leq q$ where q is the number of edges of $G(q = |E(G)|)$.

PROPOSITION 4: *Graph* G *is triangleless iff* $\omega^*(G) = q$.

The graph G in figure 3 gives a counterexample to the converse of Theorem 3 since $\omega(G) = \omega^*(G) = 3$ and, while G has no isolated vertices, the vertices $S_1 = \{a,b,c\}$ and $S_2 = \{a,b\}$ satisfy $N[S_1] = N[S_2]$. Consideration of edges x, y, and z show that $\theta_1(G) \geq 3$. In general, if $\epsilon(G)$ is the maximum number of edges, no two of which are in a common clique, then clearly $\theta_1(G) \geq \epsilon(G)$. (A *clique* is a maximal complete subgraph.)

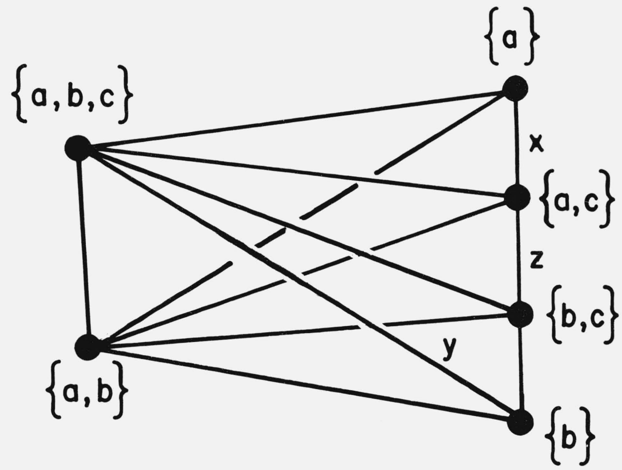

FIGURE 3. *A graph with* N[{a,b,c}] = N[{a,b}] *and* ω(G) = ω^*(G).

To evaluate $\theta_1(G)$, consider the set of complete subgraphs selected to cover $E(G)$. Since every complete subgraph is contained in a clique, $\theta_1(G)$ can be defined as the minimum number of cliques of G which contain all the edges of G. Let $C_1,C_2,\ldots,C_t$ be the cliques of G, and let V_i and E_i be the vertex and edge sets, respectively, of C_i. The *clique* graph of G, denoted $C(G)$, is the intersection graph on $V(G)$ with $F_1 = \{V_1,V_2,\ldots,V_t\}$; let the *clique-edge graph*, denoted $C(G)$, be the intersection graph on $E(G)$ with $F_2 = \{E_1,E_2,\ldots,E_t\}$. Thus $C(G)$ can be considered to be the graph whose vertices are the cliques of G, with two cliques adjacent iff they have an edge in common. If G has no isolated vertices, then it can be seen that $C(G)$ is obtained from $C(G)$ by deleting each edge corresponding to two cliques intersecting in exactly one point.

The work of Hamelink [4] and Roberts and Spencer [8] gives us necessary and sufficient conditions for a graph H to be the clique graph of some graph G. These same conditions can be shown to be necessary and sufficient for H to be the clique-edge graph of some graph F. In general, let $C_k(G)$ be the graph whose vertices are the cliques of G, with two cliques adjacent iff they have at least k vertices in common. Given H then there is a graph F with $H = C_k(F)$ iff there is a graph G with $H = C(G)$ (See [8].)

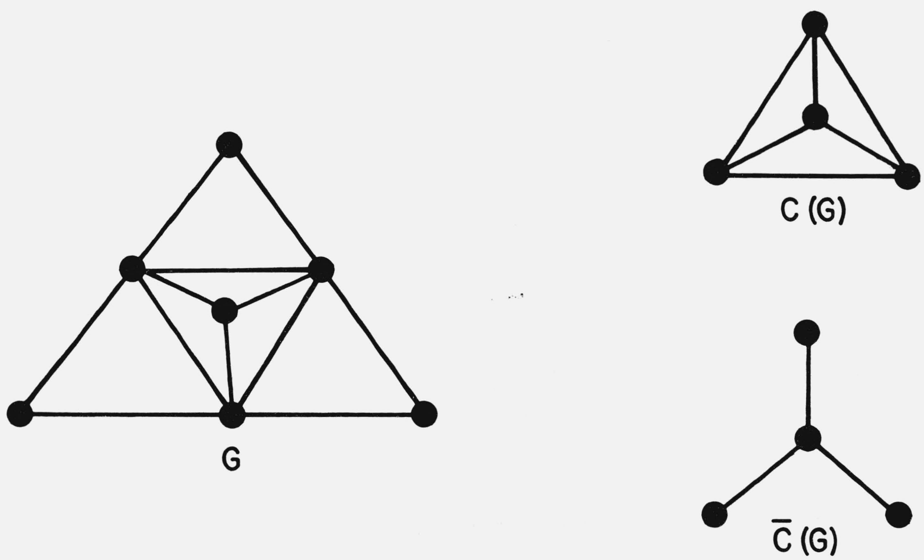

FIGURE 4. *A graph G and its clique graphs.*

In finding $\theta_1(G)$ one has a collection of subsets of $E(G)$, namely $\{E_1, E_2, \ldots, E_t\}$, and one needs to select a subcollection with the smallest number of elements that still covers $E(G)$. Thus evaluating the pseudointersection number of G is a set covering problem. Much work has been done on set covering problems using integer programming, for example, by Garfinkel and Nemhauser [2, chapter 8]. The "reductions" possible for set covering problems lead to bounds for $\theta_1(G)$.

For example, consider the following. Let p_i be the vertex of $C(G)$ corresponding to E_i for $1 \leq i \leq t$. Suppose that there is an edge e_j in E_j such that $e_j \epsilon E_i$ when $j \neq i (1 \leq i \leq t)$ iff $1 \leq i \leq k$. Let $a_0(G)$ denote the minimum number of vertices of graph G such that every edge of G is incident with at least one of these vertices.

PROPOSITION 5: $k \leq \theta_1(G) = \omega^*(G) \leq k + a_0\Big(C(G) - \{p_1, \ldots, p_k\}\Big)$.

PROOF: Suppose $E_i (1 \leq i \leq t)$ and $e_j (1 \leq j \leq k)$ are as described above. Clearly E_1, $E_2, \ldots, E_k$ must be chosen to cover $e_1, e_2, \ldots, e_k$, and so $k \leq \theta_1(G)$. Now for any edge e of G in two or more cliques there are two or more adjacent points of $C(G)$, say $u_1, u_2, \ldots$. If one of these is a $p_j (1 \leq j \leq k)$, then e is in the clique E_j already selected. If not, then edge (u_1, u_2) is in $C(G) - \{p_1, \ldots, p_k\}$. Consequently $C_1, \ldots, C_k$ and the cliques corresponding to an a_0 set of $C(G) - \{p_1, \ldots, p_k\}$ cover all the edges of G.

3. Observations

In addition to the advantages of pseudorepresentations over representations obtained by a direct concentration on adjacency requirements, there are also situations in which pseudointersection graphs can be formed while intersection graphs cannot. Given any m by n 0, 1-matrix M, one may, for example, form a pseudointersection graph on m vertices $v_1, v_2, \ldots, v_m$ by using a pseudorepresentation $r: \{v_1, \ldots, v_m\} \rightarrow \{0,1\}^n$ where $r(v_i)$ is the ith row of M. Very often r will not be a representation. Likewise one can reverse the rows and columns (that is, treat M^T as above). As an example, one obtains the line graph of G if the transpose of the incidence matrix is used.

Often one forms a graph H from a given graph G by a fixed procedure, such as forming the line graph or clique graph. Much recent work has been done to investigate what happens when the operation is iterated, for example in [1] and [6]. Such iterations on the formation of pseudointersection graphs will be examined in [9].

4. References

[1.] Dongre, N. M., On the squares of cycles and other graphs, Indian Journal of Pure and Appl. Math. **4,** 1–4 (1973).
[2.] Garfinkel, R. S., and Nemhauser, G. L., Integer Programming, (John Wiley and Sons, New York, 1972).
[3.] Harary, F., Graph Theory (Addison-Wesley, Reading, Mass., 1969).
[4.] Hamelink, R. C., A partial characterization of clique graphs, Journal of Combinatorial Theory **5,** 192–197 (1968).
[5.] Hedetniemi, S. T. personal communication.
[6.] Hedetniemi, S. T., and Slater, P.J., Line graphs of triangleless graphs and iterated clique graphs, in Graph Theory and Applications (Springer-Verlag, Berlin, 1972), pp. 139–149.
[7.] Marczewski, E., Sur deux proprietes des classes d'ensembles, Fund. Math. **33,** 303–307 (1945).
[8.] Roberts, F. S., and Spencer, J. H., A characterization of clique graphs, Journal of Combinatorial Theory **10,** 102–108 (1971).
[9.] Slater, P. J., Pseudointersection operators, in preparation.

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