JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 80B, No. 3, July–September 1976

Sampling Expansion for a Languerre- L_ν^α Transform*

A. J. Jerri**

(June 14, 1976)

The Whittaker cardinal (or sampling) series expansion, which is associated with a finite limit Fourier transform, has been extended to include other kernels that are solutions of the Sturm-Liouville problem. To date all illustrations of this generalized sampling expansion have been associated with orthogonal functions on finite intervals. In this note, we present a sampling expansion for a Laguerre-L_ν^α transform where we shall use the Laguerre polynomials $L_n^\alpha(x)$ which are orthogonal on the semi-infinite interval $(0, \infty)$ with respect to the weight function $e^{-x}x^\alpha$.

Key words: Laguerre transform; sampling function; Whittaker's cardinal function.

1. Introduction

Let I be an interval on $(-\infty, \infty)$, let $\rho \geq 0$ on I, and let $L^2(I; \rho)$ denote the family of all functions defined on I such that

$$\int_I \rho(x)|F(x)|^2 dx < \infty.$$

Let $K(t,\cdot) \epsilon L^2(I; \rho)$ for each $t \epsilon I$, and let $B(I; \rho)$ denote the family of all functions f such that

$$f(t) = \int_I \rho(x) F(x) K(t, x) dx, \ t \epsilon I, \tag{1}$$

where $F \epsilon L^2(I; \rho)$. The sampling theorem states that if there exists a countable set $\{t_n\}$ such that $\{K(t_n, \cdot)\}$ is a complete orthogonal set on $L^2(I; \rho)$ then every $f \epsilon B(I; \rho)$ has the representation

$$f(t) = \lim_{N \to \infty} \sum_{|n| \leqq N} f(t_n) S_n(t) \tag{2}$$

where

$$S_n(t) = \frac{\displaystyle\int_I \rho(x) K(t, x) \overline{K(t_n, x)} dx}{\displaystyle\int_I \rho(x) |K(t_n, x)|^2 dx}. \tag{3}$$

Here $\overline{K(t_n, x)}$ is the complex conjugate of $K(t_n, x)$, $S_n(t)$ is termed the sampling function and $f(t_n)$ are obviously the samples of $f(t)$ at t_n.

This sampling expansion (2) was illustrated for the kernel $K(t, x)$ as Bessel functions [1], the Legendre functions [2]; and for the associated Legendre functions, the Gegenbauer functions, a Chebyshev function, the prolate spheroidal functions, and a Bessel like function satisfying a fourth order differential equation [3]. All these illustrations are associated with a finite limit integral transform (1) and hence the use of orthogonal functions on this finite interval I.

Next we present an illustration for a Laguerre-L_ν^α transform where we will use the Laguerre polynomials L_n^α which are orthogonal on the semi-infinite interval $(0, \infty)$.

---

*An invited paper.
**Present address: Department of Mathematics, Clarkson College of Technology, Potsdam, New York 13676.

2. Sampling for a Laguerre-L_ν^α Transform

In this section we shall use the notation of the previous section. Specifically, we shall take

$$I=(0,\ \infty),\ \rho(x)=e^{-x}x^\alpha,$$

and

$$K(\nu, x)=L_\nu^\alpha\left(\frac{\lambda x}{\lambda-1}\right)$$

where $\alpha>-1$, $\nu\geqq 0$ and $-1<\lambda<1$, $\lambda\neq 0$. Consider the function $F(x)\epsilon L^2(I;\ \rho)$ with $\rho(x)=e^{-x}x^\alpha$ and its following Laguerre transform,

$$f(\nu)=\int_0^\infty e^{-x}x^\alpha L_\nu^\alpha\left(\frac{\lambda x}{\lambda-1}\right)F(x)dx, \qquad \alpha>-1,\ \nu\geq 0 \tag{4}$$

with the so-called Laguerre function,

$$L_\nu^\alpha(x)=\frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)\Gamma(\alpha+1)}M(-\nu,\alpha+1;x) \tag{5}$$

where $M(a,\ b;\ x)$ is the confluent hypergeometric function. We should point out here that the Laguerre function in (5) is defined differently from that of [4, p. 268, eq (37)],

$$L_\nu^\alpha(x)=\frac{1}{\Gamma(\nu+1)}M(-\nu,\alpha+1;x) \tag{6}$$

but it reduces to the Laguerre polynomial $L_n^\alpha(x)$ when $v=n$. The sampling expansion for the Laguerre transform (4) is

$$f(\nu)=\frac{1}{(1-\lambda)^\nu\Gamma(\nu+1)}\left[f(0)+\lim_{N\to\infty}\sum_{n=1}^{N}(-\nu)_n\sum_{m=0}^{n}\frac{(\lambda-1)^{n-m}f(n-m)}{m!}\right]. \tag{7}$$

We note here that in contrast to the other sampling expansions which involve the nth sample $f(n)$ in the nth term of the sampling series, the sampling expansion in (7) involves a combination of the first $n+1$ samples of the function in the nth term of the sampling series. However for $v=k$, a nonnegative integer, the sampling expansion (7) gives the sample values $f(k)$. To verify this we note that the summation over n in (7) stops at $n=k$ and all the coefficients of $(\lambda-1)^{n-m}$ in the double series cancel out, except that of $(\lambda-1)^k$ which reduces (7) to $f(k)$. To prove (7) we first write the $L_n^\alpha(x)$-Laguerre polynomials orthogonal expansion for $F(x)$ in (4) and formally integrate term by term to obtain,

$$\begin{aligned} f(\nu)&=\sum_{n=0}^{\infty}c_n\int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)L_\nu^\alpha\left(\frac{\lambda x}{\lambda-1}\right)dx \\ &=\frac{-\Gamma(\alpha+1)\sin\pi\nu}{\pi(1-\lambda)^\nu}\sum_{n=0}^{\infty}\frac{c_n\lambda^n}{n!}\Gamma(-\nu+n) \end{aligned} \tag{8}$$

after using [5, p. 75, eq. (204)],

$$\int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)M\left(c,\alpha+1;\frac{\lambda x}{\lambda-1}\right)dx=\lambda^n\frac{(1-\lambda)^c\Gamma(c+n)\Gamma(\alpha+1)}{n!\Gamma(c)} \tag{9}$$

and

$$\Gamma(-\nu)\Gamma(\nu+1)=-\frac{\pi}{\sin\pi\nu} \tag{10}$$

and where

$$c_n=\frac{n!}{\Gamma(n+\alpha+1)}\int_0^\infty e^{-x}x^\alpha F(x)L_n^\alpha(x)dx. \tag{11}$$

A more rigorous proof of (8) for $F(x)\epsilon L^2(0, \infty)$, with respect to $\rho(x)=e^{-x}x^{\alpha}$, involves using the generalized Parseval's equality for Laguerre orthogonal expansion,

$$\int_0^\infty e^{-x}x^{\alpha}f(x)g(x)dx=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha+1)}{n!}c_n d_n \tag{12}$$

for $f(x)$, $g(x)\epsilon L^2(I, \rho)$ and where c_n and d_n are the Fourier coefficients of $f(x)$ and $g(x)$, respectively. In (8), we have $f(x)=F(x)\epsilon L^2(I, \rho)$ and from the asymptotic expansion we can show that

$$g(x)=L_{\nu}^{\alpha}\left(\frac{\lambda x}{\lambda-1}\right)\epsilon L^2(I; \rho)$$

where

$$\frac{\lambda}{\lambda-1}<\frac{1}{2}$$

The cofficients d_n and c_n are obtained from (9) and (11), respectively. The series in (12) converges absolutely since according to Parseval's equation, with $f(x)$, $g(x)\epsilon L^2(I, \rho)$, both

$$\left[\frac{\Gamma(\nu+\alpha+1)}{n!}\right]^{1/2}\cdot c_n \text{ and } \left[\frac{\Gamma(\nu+\alpha+1)}{n!}\right]^{1/2}\cdot d_n$$

are square summable and hence according to Cauchy's inequality the series in (12) coverges absolutely.

For the coefficients c_n in (11), we note that the integral expression for c_n is not in the form of the Laguerre transform (4), in order to represent the samples $f(n)$, as in all the previous sampling expansions [1, 2, 3]. To express c_n in terms of the samples of $f(\nu)$ we shall use the following identity [6, p. 192, eq. (40)],

$$L^{\alpha}{}_n(\mu x)=\sum_{m=0}^{n}\binom{n+\alpha}{m}\mu^{n-m}(1-\mu)^m L^{\alpha}_{n-m}(x) \tag{13}$$

to express $L_n{}^{\alpha}(x)$ in the integral (11), in terms of

$$L^{\alpha}{}_n\left(\frac{\lambda x}{\lambda-1}\right)$$

as follows:

$$L_n{}^{\alpha}(\mu x)=L^{\alpha}{}_n\left(\frac{\lambda x}{\lambda-1}\cdot\frac{\lambda-1}{\lambda}\right)=\frac{1}{\lambda^n}\sum_{m=0}^{n}\binom{n+\alpha}{m}(\lambda-1)^{n-m}L^{\alpha}_{n-m}\left(\frac{\lambda x}{\lambda-1}\right). \tag{14}$$

If we use this expression for $L_n{}^{\alpha}(x)$ in (11), we obtain,

$$\begin{aligned} c_n&=\frac{n!}{\lambda^n\Gamma(n+\alpha+1)}\sum_{m=0}^{n}\binom{n+\alpha}{m}(\lambda-1)^{n-m}\int_0^\infty e^{-x}x^{\alpha}F(x)L^{\alpha}_{n-m}\left(\frac{\lambda x}{\lambda-1}\right)dx \\ &=\frac{n!}{\lambda^n\Gamma(n+\alpha+1)}\sum_{m=0}^{n}\binom{n+\alpha}{m}(\lambda-1)^{n-m}\frac{\Gamma(n-m+\alpha+1)}{\Gamma(\alpha+1)}f(n-m) \end{aligned} \tag{15}$$

where we have used the identity (4) with $\nu=n-m$ to replace the integral in (15) by $f(n-m)$. The sampling expansion (7) is obtained when we substitute the extreme right-hand side of (15) for c_n in (8).

The author would like to thank the referee and the associate editor, F. W. J. Olver, of this Journal for their valuable suggestions.

215-653 O - 77 - 7

3. References

[1] Kramer, H. P., A generalized sampling theorem, J. Math Phys. **38** (1959) (68–72).

[2] Campbell, L. L., A comparison of the sampling theorems of Kramer and Whittaker, Soc. Indust. Appl. Math. **12** (1964).

[3] Jerri, A. J., On the applications of some interpolating functions in physics, J. Res. Nat. Bur. Stand. (U.S.), **73B** (Math. Sci.) No. 3, 241–245 (July–Sept. 1969).

[4] Erdélyi, A., et. al., Higher Transcendental Functions, I (McGraw–Hill Publishing Company, New York, 1953).

[5] McCully, J. C., Operational Calculus of the Laguerre Transform, Ph.D. Thesis, Department of Mathematics, Univ. of Michigan, 1957.

[6] Erdélyi, A., et. al., Higher Transcendental Functions, II (McGraw-Hill Publishing Company, New York 1953).

(Paper 80B3–453)