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Spectral Measures and Separation of Variables*

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This article gives an expression for the spectral measure corresponding to a self-adjoint operator for which separation of variables is possible. The construction makes use of the amalgamation theorem for normal operators in a natural way to obtain the required measure as a tensor convolution of the spectral measures of the part operators.

Key words: Convolution; Hilbert space; separation of variables; spectral measure; tensor products.

Let A be a self-adjoint operator in a complex Hilbert space $\mathfrak{H}$, $(\cdot, \cdot)$. A *separation of variables* consists of a description of $\mathfrak{H}$ as a Hilbert space tensor product

$$\mathfrak{H}=\mathfrak{H}_1\otimes\mathfrak{H}_2$$

of two spaces $\mathfrak{H}_1$, $(\cdot, \cdot)_1$ and $\mathfrak{H}_2$, $(\cdot, \cdot)_2$ together with a decomposition of A. The requirements of the decomposition are (1) that there exist self-adjoint operators A_1 in $\mathfrak{H}_1$ and A_2 in $\mathfrak{H}_2$ such that on the elementary products $u_1\otimes u_2$, u_1 in a core $\mathfrak{D}_1$ of A_1 and u_2 in a core $\mathfrak{D}_2$ of A_2, A has the expression

$$A(u_1\otimes u_2)=A_1u_1\otimes u_2+u_1\otimes A_2u_2;$$

and (2) that the linear hull $\mathfrak{D}$ of such products be a core of A. The operator A is said to be *separated* with A_1 and A_2 *part operators*, and the decomposition is written

$$A=A_1\otimes I_2+I_1\otimes A_2.$$

Denote by E, E_1 and E_2 the spectral measures corresponding to A, A_1 and A_2, respectively. The goal is to give meaning to and to justify the relation

$$E=E_1 \circledast E_2.$$

The first steps are to use the amalgamation theorem to define a *tensor product spectral measure* $E_1\otimes E_2$ analogous to the product measure for complex measures. Then the *tensor convolution* $E_1 \circledast E_2$ has a natural definition. Finally, $E_1 \circledast E_2$ is identified with E.

Denote by $\mathfrak{B}$ the family of Borel sets of the reals R, and by $\mathfrak{B}^2$ the Borel sets of $R\times R$. A spectral measure defined over $\mathfrak{B}$ is said to be real. Here all spectral measures are normalized. The generality needed here is given by the following version of the

AMALGAMATION THEOREM: *If $\hat{E}_1$ and $\hat{E}_2$ are commuting real spectral measures in a Hilbert space $\mathfrak{H}$, then there exists one and only one spectral measure $\hat{E}$ in $\mathfrak{H}$ over $\mathfrak{B}^2$ such that*

$$\hat{E}(B\times B')=\hat{E}_1(B)\hat{E}_2(B'), \qquad \textit{for all } B,\ B'\in\mathfrak{B}.$$

That $\hat{E}_1$ and $\hat{E}_2$ commute means

$$\hat{E}_1(B)\ \hat{E}_2(B')=\hat{E}_2(B')\ \hat{E}_1(B)$$

for all B $B'\in\mathfrak{B}$.

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The problem at hand is to use the amalgamation theorem to construct the spectral measure E for a separating self-adjoint operator A. The first step is

LEMMA 1: *If E_1 and E_2 are real spectral measures in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, then $E_1 \otimes I_2$ and $I_1 \otimes E_2$ are commuting real spectral measures in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.*

PROOF: Clearly $E_1 \otimes I_2$ defined by

$$(E_1 \otimes I_2)\,(B) = E_1\,(B) \otimes I_2, \qquad \text{for all } B \epsilon\, \mathfrak{B},$$

is an $\mathfrak{H}_1 \otimes \mathfrak{H}_2$-projection-valued function defined on $\mathfrak{B}$, and

$$(E_1 \otimes I_2)\,(R) = I_1 \otimes I_2 = I.$$

The countable additivity follows from that of E_1 and the fact that strong convergence of factor operators implies strong convergence of their tensor product. Thus $E_1 \otimes I_2$ is a real spectral measure; and by a parallel argument, so is $I_1 \otimes E_2$. Finally

$$[(E_1 \otimes I_2)\,(B)]\,[(I_1 \otimes E_2)\,(B')] = E_1(B) \otimes E_2(B') = [(I_1 \otimes E_2)\,(B')]\,[(E_1 \otimes I_2)\,(B)], \text{ for all } B,\ B' \epsilon\, \mathfrak{B},$$

by elementary computations.

The next step is to establish a product spectral measure analogous to a product measure derived from ordinary measures. As usual, the product is defined on rectangles and then extended. This matter is taken care of by

LEMMA 2: *The $\mathfrak{H}_1 \otimes \mathfrak{H}_2$-projection-valued set function $E_1 \otimes E_2$ defined on rectangles $B \times B' \in \mathfrak{B}^2$* by

$$(E_1 \otimes E_2)\,(B \times B') = E_1(B) \otimes E_2(B')$$

has an unique extension as a spectral measure in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ over $\mathfrak{B}^2$.

PROOF: This is a direct application of the amalgamation theorem in which

$$\hat{E}_1 = E_1 \otimes I_2, \ \hat{E}_2 = I_1 \otimes E_2,$$

and

$$\hat{E} = E_1 \otimes E_2.$$

The relation

$$\hat{E}(B \times B') = \hat{E}_1(B)\hat{E}_2(B')$$

can be read off from the last lines of the proof of Lemma 1.

It is quite natural to call $E_1 \otimes E_2$ the *tensor product spectral measure* of E_1 with E_2.

The third step is to define the tensor convolution of E_1 with E_2 as for convolutions of complex measures. In preparation we need

LEMMA 3: *Let E be a spectral measure on R^2 and for each $B \in \mathfrak{B}$ let*

$$B^2(B) = \{(x, y) \in R^2 | x+y \in B\},$$

then E_* *defined on $\mathfrak{B}$ by*

$$E_*(B) = E[B^2(B)]$$

is a real spectral measure.

PROOF: Since

$$B^2(B) \in \mathfrak{B}^2, \qquad \text{for all } B \in \mathfrak{B},$$

E_* is a projection-valued set function on $\mathfrak{B}$; and clearly

$$E_*(R) = I.$$

Further if

$$B \cap B' = \emptyset,$$

then

$$B^2(B) \cap B^2(B') = \emptyset,$$

and if

$$B = \bigcup_i B_i$$

then

$$B^2(B) = \bigcup_i B^2(B_i).$$

These follow directly from the definition of $B^2(B)$. Hence if

$$B = \bigcup_i B_i$$

and

$$B_i \cap B_j = \emptyset,\ i \neq j,$$

then

$$E_*(B) = E[B^2(B)] = E[\bigcup_i B^2(B_i)] = \sum_i E[B^2(B_i)] = \sum_i E_*(B_i),$$

so that E_* is countably additive.

Now it is natural to formulate the

DEFINITION: *Let E_1 and E_2 be real spectral measure in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, and let $E_1 \otimes E_2$ be their tensor product. The tensor convolution of E_1 with E_2, designated $E_1 \circledast E_2$, is the real spectral measure in $\mathfrak{H}_1 \circledast \mathfrak{H}_2$ given by*

$$(E_1 \circledast E_2)(B) = (E_1 \otimes E_2)[B^2(B)], \qquad \text{for all } B \in \mathfrak{B},$$

where B²(B) *is as defined in Lemma* 3.

The tensor convolution of E_1 with E_2 has a tidy relation to the convolution of the measures associated with E_1 and E_2 as given by

LEMMA 4: *Let $E_* = E_1 \circledast E_2$, where E_1 and E_2 are real spectral measures in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, and let $u = u_1 \circledast u_2$ and $v = v_1 \circledast v_2$ be elementary tensor products, then for all such u and v*

$$(E_* u, v) = (E_1 u_1, v_1)_1 \circledast (E_2 u_2, v_2)_2,$$

where $\circledast$ indicates the convolution of measures.

PROOF: Using the definition of $E_1 \circledast E_2$ it is evident that

$$((E_1 \otimes E_2)(B \times B')u, v) = (E_1(B)u_1, v_1)_1 (E_2(B')u_2, v_2)_2, \text{ for all } B,\ B' \in \mathfrak{B}.$$

Since the product measure

$$(E_1 u_1, v_1)_1 \times (E_2 u_2, v_2)_2,$$

is the unique extension to $\mathfrak{B}^2$ of the right side of the preceding equation and $((E_1 \otimes E_2)u, v)$ is also an extension to $\mathfrak{B}^2$, the two extensions coincide, i.e.,

$$((E_1 \otimes E_2)\ (B^2)u, v) = [(E_1 u_1, v_1)_1 \times (E_2 u_2, v_2)_2]\ (B^2), \qquad \text{for all } B^2 \in \mathfrak{B}^2.$$

On specializing this to $B^2(B)$ for any $B \in \mathfrak{B}$ and invoking the definitions of the convolutions, the desired result follows.

Based on what has been done so far, it is now quite easy to show that

$$E_* = E_1 \circledast E_2$$

is, indeed, the spectral measure corresponding to A. From Lemma 3 and the Definition, it is clear that E_* is a real spectral measure and consequently corresponds to some self-adjoint

215-653 O - 77 - 2

operator A_* in $\mathfrak{H}$. The business at hand is to show that A_* is equal to A. This will complete the construction and prove the

THEOREM: *Let A be a separated self-adjoint operator with A_1 and A_2 part operators, and let E_1 and E_2 be the real spectral measures corresponding to A_1 and A_2, respectively; then the real spectral measure given by the tensor convolution $E_1 * E_2$ of E_1 with E_2 corresponds to A.*

PROOF: Let A_* be the self-adjoint operator corresponding to E_*. We shall show that A_* is defined and coincides with A on the core $\mathfrak{D}$ of A made up of finite linear combinations of elementary products $u_1 \otimes u_2$, $u_1 \in \mathfrak{D}_1$, $u_2 \in \mathfrak{D}_2$.

Recall that the domain $\mathfrak{D}_*$ of A_* is given by

$$\mathfrak{D}_* = \{u \in \mathfrak{H} | \int \lambda^2 d(E_* u, u) < \infty\}$$

and that

$$(A_* u, v) = \int \lambda d(E_* u, v)$$

for all $u \in \mathfrak{D}_*$, $v \in \mathfrak{H}$. Let $u = u_1 \otimes u_2$, $u_1 \in \mathfrak{D}_1$, $u_2 \in \mathfrak{D}_2$ and let $v = v_1 \otimes v_2$.

By separation of variables and the spectral theorem for A_1 and A_2 it follows that

$$||Au||^2 = \iint_{\mathrm{R}^2} (\lambda_1 + \lambda_2)^2 d(E_1 u_1, u_1)_1 d(E_2 u_2, u_2)_2$$

and

$$(Au, v) = \iint_{\mathrm{R}^2} (\lambda_1 + \lambda_2) d(E_1 u_1, v_1)_1 d(E_2 u_2, v_2)_2.$$

By Fubini's theorem

$$||Au||^2 = \int_{\mathrm{R}^2} (\lambda_1 + \lambda_2)^2 d[(E_1 u_1, u_1)_1 \times (E_2 u_2, u_2)_2]$$

and

$$(Au, v) = \int_{\mathrm{R}^2} (\lambda_1 + \lambda_2) d[(E_1 u_1, v_1)_1 \times (E_2 u_2, v_2)_2].$$

Now by an immediate consequence of the definition of convolution of measures

$$||Au||^2 = \int_{\mathrm{R}} \lambda^2 d[(E_1 u_1, u_1)_1 \circledast (E_2 u_2, u_2)_2]$$

and

$$(Au, v) = \int_R \lambda d[(E_1 u_1, v_1)_1 \circledast (E_2 u_2, v_2)_2].$$

According to Lemma 4, this is the same as

$$||Au||^2 = \int_R \lambda^2 d(E_1 \circledast E_2 u, u)$$

and

$$(Au, v) = \int_{R^1} \lambda d(E_1 \circledast E_2 u, v).$$

Thus A_* is defined on each such u, and by linearity on $\mathfrak{D}$. Further, by the last equation

$$(Au, v)=(A_*u, v) \qquad \text{for all } u\in \mathfrak{D}, \qquad v=v_1\otimes v_2.$$

But since elementary products are total in $\mathfrak{H}$,

$$Au=A_*u, \qquad \text{for all } u\in \mathfrak{D},$$

as was to be shown.

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Paper 80B3–450