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One-Sided Tolerance Limits for the Normal Distribution, $P=0.80$, $\gamma=0.80$

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A table is given of factors k used in constructing one-sided tolerance limits for a normal distribution. This table was obtained by interpolation in an existing table of percentage points of the noncentral t-distribution. The accuracy of the table is estimated, and a comparison is made of the presently computed factors with a previously published approximation.

Key words: Noncentral t-distribution; normal distribution; statistics; tolerance limits.

1. Introduction

Let X be a normal random variable with mean μ and standard deviation σ. If μ and σ are known, we can say that exactly a proportion P of the population is below $\mu+K_p\sigma$, where K_p is a normal deviate defined by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{K_p} \exp(-t^2/2)dt=P. \tag{1}$$

If μ and σ are unknown, one can estimate these quantities from a random sample of n observations: $x_1, x_2, \ldots, x_n$. The mean μ is estimated by

$$\bar{x}=\frac{1}{n}\sum_{i=1}^{n} x_i,$$

and the standard deviation σ is estimated by

$$s=\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}.$$

The problem is now to find k such that the probability is γ that at least a proportion P of the population is below $\bar{x}+ks$. Mathematically, the problem is to find k such that

$$Pr\{Pr(X\leq\bar{x}+ks)\geq P\}=\gamma \tag{2}$$

where X has a normal distribution with mean μ and standard deviation σ, and P and γ are specified probabilities. As is indicated in Owen [10], eq (2) can be written as

$$Pr\{T_f\leq k\sqrt{n} \mid \delta=K_p\sqrt{n}\}=\gamma$$

where T_f denotes the noncentral t-distribution with $f=n-1$ degrees of freedom and with noncentrality parameter

$$\delta=K_p\sqrt{n}.$$

2. Existing Tables

For many specified values of P and γ, tables of factors k for one-sided tolerance limits as defined in (2) above have been published. References to these tables are listed in Owen [10] and Johnson and Kotz [4], chapter 31. Since none of the tables cited in these references gives exact values of k in the case where $P=0.80$ and $\gamma=0.80$, the present table has been prepared to fill this gap. Approximations to the factors k for this case were included in table III of Owen [8].

3. Computing Method for the Present Table

Exact values of k can be computed from the appropriate percentage points of the noncentral t-distribution. Table III (pp. 214–237) of Locks, Alexander, and Byars [7] gives 3-decimal percentage points of the noncentral t for:

$$f=n-1=1(1)20, 25, 30, 35, 40; K_p=0.00(0.25)3.00; \epsilon=1-\gamma=0.01, 0.05(0.05)0.95, 0.99, 0.995.$$

By interpolating on the values in this table for $\epsilon=0.20$, one can obtain the factors k corresponding to $P=0.80$, $\gamma=0.80$.

Table 1 presented here gives one-sided tolerance factors k, Owen's approximate values of k and the relative error in the approximate values, for $n=2(1)21$, 26, 31, 36, 41. The values of k were obtained from the 3-decimal percentage points of the noncentral t-distribution given by Locks et al. [7], using five-point Lagrangian interpolation. Four-point Lagrangian interpolation yields the same values of k (to 3 decimals) as does five-point interpolation. These computations were done through the use of OMNITAB (Hogben et al. [2]). For $n=4(1)12$, the values of k were also computed using the first five terms of Stirling's interpolation formula as given on page 71 of Kunz [6]; again the same values of k (to 3 decimals) were obtained. For all these calculations the value of $K_p=K_{0.80}=0.84162123$ was taken from a table of the inverse normal probability distribution (Kelley [5]).

TABLE 1. *One-sided tolerance limit factors for the normal distribution*

Values of k such that $Pr\{Pr(X\leq\bar{x}+ks)\geq P\}=\gamma$ for $P=0.80$, $\gamma=0.80$

n	k	Approx. k	Rel. error
2	3.420	2.37544	0.305
3	2.016	1.70985	.152
4	1.675^{-}	1.50952	.099
5	1.514	1.40392	.073
6	1.417	1.33609	.057
7	1.352	1.28781	.047
8	1.304	1.25119	.040
9	1.266	1.22219	.035
10	1.237	1.19849	.031
11	1.212	1.17866	.028
12	1.192	1.16175	.025
13	1.174	1.14711	.023
14	1.159	1.13427	.021
15	1.145^{+}	1.12290	.019
16	1.133	1.11274	.018
17	1.123	1.10358	.017
18	1.113	1.09528	.016
19	1.104	1.08771	.015
20	1.096	1.08076	.014
21	1.089	1.07436	.013
26	1.060	1.04855	.011
31	1.039	1.02968	.009
36	1.023	1.01512	.008
41	1.010	1.00346	.007

4. Estimated Accuracy

Locks et al. [7] reported that numerous checks were made of their tables against previously published tables, and "in no case where comparison was made . . . is the disagreement more than one unit in the last decimal place."

In order to assess the accuracy of the k's given here in table 1 for $P=0.80$, $\gamma=0.80$, the following checks were made. Starting with the 3-decimal percentage points in table III of Locks for $\gamma=0.75$ and 0.90, values of k for all possible combinations of $\gamma=0.75$, $\gamma=0.90$, $P=0.75$, $P=0.90$, and $n=2(1)21(5)41$ were computed by five-point Lagrangian interpolation as described in section 3 above. These values were then compared with the exact 3-decimal values published by Owen [9], pp. 52 and 58. This permitted the comparison of 96 values of k. Of the 92 values corresponding to $n>2$, 88 were in full (3-decimal) agreement, and 4 differed by 0.001. For $n=2$, larger discrepancies were found. The values of k obtained here differed from Owen's exact values as indicated below:

n	P	γ	Interpolation in Locks: (A)	Owen's Exact k: (B)	Difference: (A)−(B)
2	0. 90	0. 90	10. 264	10. 253	0. 011
2	. 90	. 75	3. 994	3. 992	. 002
2	. 75	. 90	5. 859	5. 842	. 017
2	. 75	. 75	2. 227	2. 225	. 002

One may infer from these comparisons that the value of k for $P=0.80$, $\gamma=0.80$, $n=2$ given in table 1 is probably not in error by more than about 0.017 and is probably larger than the exact value. For $n>2$, any errors in the computed values of k are probably in the neighborhood of 0.001.

5. Comparison With an Approximation

The approximate values of k given here in table 1 are taken from table III of Owen [8]. This approximation was derived by Jennett and Welch [3] and was further discussed in chapter 1 of Eisenhart, Hastay and Wallis [1]. The formula for this approximation is

$$k \doteq \frac{K_p+\sqrt{K_p^2-ab}}{a}$$

where

$$a=1-\frac{K_\gamma^2}{2(n-1)},\quad b=K_p^2-\frac{K_\gamma^2}{n},$$

and K_γ is defined in the same manner as was K_p in eq (1).

The relative error in the approximate values of k shown in table 1 was computed from the formula

$$\text{Rel. error}=\left|\frac{(\text{approx. }k)-k}{k}\right|.$$

We note that in all cases covered by this table the approximations are smaller than the values of k computed by the method described in section 3. This is in agreement with Owen's statement [8] that the approximation will probably underestimate k for $\gamma \leq 0.95$.

6. References

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