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A Note on Pairs of Matrices With Product Zero

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Pairs of not necessarily hermitian matrices for which $AB=BA=0$ are characterized in terms of the singular values of $A+B$. This provides a generalization and a simpler proof of a classical result on the independence of quadratic forms in normal random variables.

Key words: Eigenvalues; independence; quadratic form; random variable; singular value decomposition.

Suppose that

$$x^T=(x_1, \ldots, x_n)$$

is a vector of independent normal 0, 1 random variables. In [3] [1] it is pointed out that

$$y_1=x^TAx \text{ and } y_2=x^TBx$$

are independent random variables, where A and B are real symmetric matrices, if and only if $AB=0$. An explanation of this fundamental fact is given in [5] and amounts to a somewhat lengthy derivation of the following:

THEOREM I: *If* A *and* B *are real symmetric matrices with eigenvalues* $\{\lambda_1, \ldots, \lambda_r, 0, \ldots, 0\}$ *and* $\{\lambda_{r+1}, \ldots, \lambda_n, 0, \ldots, 0\}$ *respectively, then* A+B *has eigenvalues* $\{\lambda_1, \ldots, \lambda_n\}$ *if and only if* AB=0.

The independence of y_1 and y_2 is, in a straightforward way, equivalent to $A+B$ having eigenvalues $\lambda_1 \ldots, \lambda_n$, and Theorem I (which was first noted by Craig [3]) is sufficiently fundamental that generally it is now at least stated in advanced texts. For example, a portion of a proof is given in [4]. Apparently in ignorance of [3, 4, 5], an alternate proof of Theorem I is given in [1].

Our goal is to give a generalization of Theorem I whose proof is quite simple. In addition to including a rather different proof of Theorem I, our observation points out that the symmetry of A and B is not an essential assumption. We recall that the singular values of a general complex matrix A are, by definition, the nonnegative square roots of the eigenvalues of A^*A. A good general reference on the singular values decomposition of a matrix is [6].

THEOREM II: *Suppose* A *and* B *are* n-by-n *complex matrices with singular values* $\{d_1, \ldots, d_r, 0, \ldots, 0\}$ *and* $\{d_{r+1}, \ldots, d_n, 0, \ldots, 0\}$ *respectively. Then* $A+B$ *has singular values* $\{d_1, \ldots, d_n\}$ *if and only if* AB=BA=0.

PROOF: We assume, without loss of generality, that $d_1, \ldots, d_n$ are nonzero and that

$$A=\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$D_1=\operatorname{diag}\{d_1, \ldots, d_r\}.$$

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[1] Figures in brackets indicate literature references at the end of this paper.

Let

$$B = U^* \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix} V$$

be a singular value decomposition of B, where

$$D_2 = \operatorname{diag}\{d_{r+1}, \ldots, d_n\}$$

and where the unitary matrices U and V are partitioned

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}, \quad V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$$

conformally with D_1 and D_2. First, suppose that $A+B$ has singular values $\{d_1, \ldots, d_n\}$. We then have

$$\begin{aligned}
d_1 d_2 \ldots d_n &= |det(A+B)| \\
&= \left| \det \left[\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} + U^* \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix} V \right] \right| \\
&= \left| \det \left[U \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix} V \right] \right| \\
&= \left| \det \left[\begin{pmatrix} U_1 D_1 & 0 \\ U_3 D_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ D_2 V_3 & D_2 V_4 \end{pmatrix} \right] \right|
\end{aligned}$$

$$= |det\ U_1 D_1\ det\ D_2 V_4| = d_1 d_2 \ldots d_n |det\ U_1 \det V_4|.$$

This implies that $|det\ U_1\ det\ V_4| = 1$ which, because of Hadamard's determinantal inequality, in turn implies that U_1 and V_4 are both unitary. This means that U_2, U_3, V_2 and V_3 are all 0, that

$$B = \begin{pmatrix} 0 & 0 \\ 0 & U^*_4\ D_2\ V_4 \end{pmatrix},$$

and that $AB = BA = 0$.

On the other hand, suppose that $AB = BA = 0$. If B is partitioned,

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

as U and V, then $AB=0$ implies that $B_1=0$ and $B_2=0$, while $BA=0$ implies that $B_1=0$ and $B_3=0$. Thus, B_4 has singular values $\{d_{r+1}, \ldots, d_n\}$, and

$$A+B = \begin{pmatrix} D_1 & 0 \\ 0 & B_4 \end{pmatrix}$$

has singular values $\{d_1, \ldots, d_n\}$ as was to be shown.

Of course, for hermitian matrices $AB=0$ is equivalent to $BA=0$. However, for nonhermitian matrices, $AB=0$ does not mean that $BA=0$ so that the condition $AB=BA=0$ may not be weakened in theorem II. It should, however, be noted that, as can be easily seen via the proof of theorem II, the assumption $AB=BA=0$ implies $(A+A^*)(B+B^*)=0$.

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