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Predictable Regular Continued Cotangent Expansions*

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Expansions of the form $x = \cot(\text{arc cot } n_0 - \text{arc cot } n_1 + \text{arc cot } n_2 - \ldots)$ are discussed. It is shown that if x is of the form $\frac{1}{2}(c + \sqrt{c^2 + 4})$, then the n's are predictable by a simple recurrence. Continued fractions derived from the expansion of x are also given.

Key words: Continued cotangent; continued fraction; quadratic irrational.

1. Introduction

In "A Cotangent Analogue of Continued Fractions" [1][1], D. H. Lehmer discussed expansions of the form

$$x = \cot(\text{arc cot } n_0 - \text{arc cot } n_1 + \text{arc cot } n_2 - \ldots). \tag{1}$$

This expansion is called a *continued cotangent*. The expansion is called a *regular continued cotangent* if

(a) n_s is a positive integer satisfying $n_s \geq n_{s-1}^2 + n_{s-1} + 1$ $(s = 1, 2, \ldots)$.

(b) If the expansion (1) is finite and n_k is the last n, then $n_k > n + n_{k-1}^2 + n_{k-1} + 1$.

Given any positive real number x, its regular continued cotangent expansion is generated by the following algorithm:

$$x_0 = x, \qquad n_0 = [x_0] \tag{2}$$

$$x_{s+1} = \frac{x_s n_s + 1}{x_s - n_s}, \; n_{s+1} = [x_{s+1}] \qquad (s = 1, 2, \ldots). \tag{3}$$

As usual, the brackets denote the greatest integer function.

Lehmer called the x_s's *complete cotangents* and the n_s's *incomplete* or *partial cotangents*.

He did not find any combination of familiar constants whose regular continued cotangent expansion was in any way predictable.

Here we present an infinite sequence of quadratic irrationals with the property that each member of the sequence has a predictable regular continued cotangent expansion.

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[1] Figures in brackets indicate the literature references at the end of this paper.

2. Irrationals With Predictable Expansions

We will show that, for a positive integer c,

$$\frac{c + \sqrt{c^2 + 4}}{2}$$

has a predictable regular continued cotangent expansion. More precisely, we state the following:

THEOREM: *Let* α, β *be the roots of the quadratic* $x^2 - cx - 1 = 0$, *with* c *a positive integer, and* $\alpha > \beta$. *Let* α *be expanded into its regular continued cotangent expansion, with* $n_0, n_1, n_2, \ldots$ *being the sequence of partial cotangent, and* $x_0, x_1, x_2, \ldots$ *being the sequence of complete cotangents. Then*

$$\text{(I)} \qquad x_k = \alpha^{3^k}$$
$$\text{(II)} \qquad n_k = \alpha^{3^k} + \beta^{3^k}.$$

PROOF: We start out by stating some facts about α and β:

(4) $$\alpha = \frac{c + \sqrt{c^2 + 4}}{2}, \ \beta = \frac{c - \sqrt{c^2 + 4}}{2}.$$

This follows from the definition of α and β.

(5) $$\alpha\beta = -1, \qquad \alpha + \beta = c, \qquad \alpha - \beta = \sqrt{c^2 + 4}, \ [\alpha] = c.$$

We now define the sequence V_k:

(6) $$V_k = \alpha^k + \beta^k \qquad (k \geq 0)$$

It is easily shown by induction that

(7) $$V_k = cV_{k-1} + V_{k-2} \qquad (k \geq 2).$$

We are now ready to prove the theorem. The proof of part (I) proceeds by induction.

A. Verification for $k = 0$:

$$x_0 = \alpha^{3^0} = \alpha^1 = \alpha$$
$$n_0 = \alpha^{3^0} + \beta^{3^0} = \alpha^1 + \beta^1 = c$$
$$= [x_0] = [\alpha] = c.$$

B. Assume the theorem is true for $k = s$. Then we want to show that the theorem is true for $k = s + 1$.

From (3), we have

$$x_{s+1} = \frac{x_s n_s + 1}{x_s - n_s}$$

$$= \frac{(\alpha^{3^s})(\alpha^{3^s} + \beta^{3^s}) + 1}{\alpha^{3^s} - (\alpha^{3^s} + \beta^{3^s})}$$

$$= \frac{\alpha^{2.3^5} + \alpha^{3^s}\beta^{3^s} + 1}{-\beta^{3^s}}$$

$$= \frac{\alpha^{2.3^5}}{-\beta^{3^s}} \qquad (\text{since } \alpha\beta = -1)$$

$$x_{s+1} = \alpha^{3^{s+1}} \qquad (\text{since } \beta = -1/\alpha)$$

The proof of part (I) of the theorem is now complete by induction. We complete the proof of the theorem by showing that

$$n_{s+1} = [x_{s+1}] = [\alpha^{3^{s+1}}] = \alpha^{3^{s+1}} + \beta^{3^{s+1}}$$

Since $\beta = -1/\alpha$, and $\alpha > 1$, we have

(8) $$-1 < \beta^r < 0 \qquad (r \text{ odd}, \geq 1).$$

If we add α^r to this inequality, we get

(9) $$\alpha^r - 1 < \alpha^r + \beta^r < \alpha^r \qquad (r \text{ odd}, \geq 1)$$

Now it is obvious that α^r is a quadratic irrational, so neither $\alpha^r - 1$ nor α^r are integers. Therefore, there must be exactly one integer between $\alpha^r - 1$ and α^r. Since $V_0 = \alpha^0 + \beta^0 = 2$ and $V_1 = \alpha^1 + \beta^1 = c$, we have, from (7), that $V_r = \alpha^r + \beta^r$ is an integer for $r \geq 0$. Therefore, V_r *is the integer* between $\alpha^r - 1$ and α^r, and we have

(10) $$[\alpha^r] = \alpha^r + \beta^r \qquad (r \text{ odd}, \geq 1).$$

We may now take $r = 3^{s+1}$ to get

$$n_{s+1} = [x_{s+1}] = [\alpha^{3^{s+1}}] = \alpha^{3^{s+1}} + \beta^{3^{s+1}} \qquad (s \geq 0).$$

The proof of both parts of the theorem is now complete.

In the following tables, we give the values of α and β for the first few values of c, and the values of n_k for the first few values of c and k.

TABLE 1. *Values of α and β*

c	α	β
1	$\frac{1}{2}(1 + \sqrt{5})$	$\frac{1}{2}(1 - \sqrt{5})$
2	$1 + \sqrt{2}$	$1 - \sqrt{2}$
3	$\frac{1}{2}(3 + \sqrt{13})$	$\frac{1}{2}(3 - \sqrt{13})$
4	$2 + \sqrt{5}$	$2 - \sqrt{5}$
5	$\frac{1}{2}(5 + \sqrt{29})$	$\frac{1}{2}(5 - \sqrt{29})$

TABLE 2. *Values of* n_k

$k \backslash c$	1	2	3	4	5
1	1	2	3	4	5
2	4	14	36	76	140
3	76	2786	46764	439204	2744420

3. Some Observations on n_k and x_k

First, we note that, as a special case of the theorem for $c = 1$, we have $\alpha = \phi$, the golden ratio, and $V_k = L_k$, the kth Lucas number [2]. In fact, we have

$$x_k = \phi^{3^k} \text{ and } n_k = L_{3^k}.$$

Second, from part (II) of the theorem, it is not difficult to show that

$$n_{k+1} = n_k^3 + 3n_k \qquad (k \geq 0). \tag{11}$$

We also point out that, empirically, the regular continued cotangent expansion of an "average" irrational number satisfies

$$n_{k+1} = O(n_k^2)$$

For $\alpha = \frac{1}{2}(c + \sqrt{c^2 + 4})$, however, we have, in view of (11)

$$n_{k+1} = O(n_k^3)$$

so that, in a certain sense, this group of quadratic irrationals is approximated unusually well by the continued cotangent algorithm.

Third, we observe that the regular continued cotangent for $1/\alpha = -\beta = \frac{1}{2}(-c + \sqrt{c^2 + 4})$ also is predictable. For if the regular continued cotangent expansion of x $(x > 1)$ is $n_0, n_1, n_2, \ldots$ then the expansion for $1/x$ is $0, n_0, n_1, n_2, \ldots$ From this it easily follows that the expansion for $1/\alpha$ is predictable as follows:

$$x_0 = 1/\alpha, \qquad n_0 = 0$$

$$x_k = \alpha^{3^{k-1}}, \qquad n_k = \alpha^{3^{k-1}} + \beta^{3^{k-1}} \qquad (k \geq 1).$$

We now introduce the sequence U_k, defined as follows:

$$U_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}. \tag{12}$$

The sequence U_k satisfies the same recurrence as V_k, that is,

$$U_k = cU_{k-1} + U_{k-2} \qquad (k \geq 2). \tag{13}$$

Now $x_k = \alpha^{3^k}$ is a quadratic irrational and can obviously be put in the form

$$x_k = v_k + u_k\sqrt{c^2+4} \tag{14}$$

where v_k and u_k are rational numbers. From the definitions of V_k and U_k, it is easily verified that

$$v_k = \frac{1}{2}V_{3^k} \tag{15}$$

$$u_k = \frac{1}{2}U_{3^k}. \tag{16}$$

Letting $m_k = U_{3^k}$, we have the following recurrence formula which can be verified by substitution:

$$m_{k+1} = (c^2+4)m_k^3 - 3m_k \tag{17}$$

$$v_{k+1} = 4v_k^3 + 3v_k. \tag{18}$$

$$u_{k+1} = 4(c^2+4)u_k^3 - 3u_k. \tag{19}$$

Lehmer [3] observed that

$$2u_k = (n_0^2+1)(n_1^2+1)(n_2^2+1)\ldots(n_{k-1}^2+1) \tag{20}$$

We also have

$$(v_{k+1})/v_k = (u_{k+1})/u_k + 2. \tag{21}$$

Many similar identities can be obtained.

4. Unusual Continued Fractions

We observe that the *regular* continued fractions for α and $1/\alpha$ are as follows:

$$\alpha = c + \frac{1}{c+}\,\frac{1}{c+}\,\frac{1}{c+}\,\frac{1}{c+}\,\cdots$$

$$1/\alpha = \frac{1}{c+}\,\frac{1}{c+}\,\frac{1}{c+}\,\frac{1}{c+}\,\cdots.$$

Lehmer showed that if the n_k's (the partial cotangents) for a real number x are known, then x can be expanded into the following irregular continued fraction:

$$x = n_0 + \frac{n_0^2+1}{n_1-n_0+}\,\frac{n_1^2+1}{n_2-n_1+}\,\frac{n_2^2+1}{n_3-n_2+}\,\cdots.$$

It can be shown by induction that the kth convergent to this continued fraction, p_k/q_k satisfies

$$p_k/q_k = U_e/U_{e-1} \qquad \text{where } e = (3^k+1)/2 \tag{22}$$

The first few such continued fractions are as follows:

$$\frac{1}{2}(1+\sqrt{5}) = 1 + \frac{2}{3+}\,\frac{17}{72+}\,\frac{5777}{439128+}\cdots$$

$$1+\sqrt{2} = 2 + \frac{5}{12+}\,\frac{197}{2772+}\,\frac{7761797}{21624369228+}\cdots$$

$$\frac{1}{2}(3+\sqrt{13}) = 3 + \frac{10}{33+}\,\frac{1297}{46728+}\,\frac{2186871697}{102266868085272+}\cdots.$$

These continued fractions converge extremely rapidly.

5. References

[1] Lehmer, D. H., A Cotangent Analogue of Continued Fractions, Duke Math. J. Vol. 4 (1938), pps. 323–340.
[2] Hoggatt, Verner E. Jr., Fibonacci and Lucas Numbers (1969).
[3] Lehmer, D. H., personal communication.

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