JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 80B2, April–June 1976

On Iohvidov's Proofs of the Fischer-Frobenius Theorem*

R. C. Thompson**

(March 12, 1976)

A short proof is given of theorem of Fischer and Frobenius exhibiting a conjunctive transformation mapping Toeplitz matrices onto Hankel matrices.

Key words: Hankel matrix; Toeplitz matrix.

1. Introduction

A Hankel matrix H_{n-1} is an $n \times n$ matrix with the structure

$$H_{n-1} = [s_{i+j}]_{0 \leqq i,j<n}.$$

A Toeplitz matrix T_{n-1} is an $n \times n$ matrix with the structure

$$T_{n-1} = [c_{i-j}]_{0 \leqq i,j<n}.$$

It is easy to see that Toeplitz matrices may be converted to Hankel matrices in a uniform way by a matrix multiplication. Indeed, if

$$J = \begin{bmatrix} & & & 1 \\ & \bigcirc & 1 & \\ & 1 & \bigcirc & \\ 1 & & & \end{bmatrix},$$

then JT_{n-1} is a Hankel matrix for all Toeplitz matrices T_{n-1}. This procedure, however, does not carry Hermitian Toeplitz matrices to Hermitian (i.e., real) Hankel matrices. The theorem of Fischer-Froebenius asserts that a class of transformations exist each of which uniformly carries Toeplitz matrices to Hankel matrices in such a way that Hermitian Toeplitz matrices are carried to Hermitian Hankel matrices. Recently I. C. Iohvidov has published three proofs of this result. One of these proofs is a direct but somewhat intricate calculation; it may be found on pages 211–213 of [1][1]. A second proof, to be found on page 217 of [1] and also in [2], makes a preliminary reduction to the case of positive definite Toeplitz matrices, then takes advantage of a decomposition of definite Toeplitz matrices known from the theory of the trigonometric moment problem. The third proof, in [2], avoids the reduction to the positive definite case, and uses instead a more complicated decomposition of Toeplitz matrices due to Iohvidov and Krein [3, p. 338].

The purpose of this paper is to give a short and direct proof of the Fischer-Frobenius theorem. Our proof is based on a simple decomposition of arbitrary Toeplitz matrices, for which the proof is almost a triviality and which was apparently not noticed in [1] and [2]. See equation (3). Iohvidov's techniques then may be applied to (3) to produce the desired result rapidly.

*An invited paper.
**Present address: Institute for Applied Algebra, University of California, Santa Barbara, CA 93106.
[1] Figures in brackets indicate the literature references at the end of this paper.

2. A Decomposition of Toeplitz Matrices

Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_{2n-2}$ be distinct fixed numbers on the unit circle, and set

$$V = [\epsilon_j{}^i]_{-(n-1)\leqq i\leqq n-1,\ 0\leqq j\leqq 2n-2}.$$

This square matrix V is nonsingular since after removal of a factor from each column it becomes a Vandermonde. Alternatively, a nontrivial linear relation on the rows of V quickly leads to the impossible existence of a nonzero polynomial of degree at most $2n - 2$ with $2n - 1$ distinct roots.

Let $T = [c_{i-j}]_{0\leq i,j\leq n-1}$ be an arbitrary Toeplitz matrix and take

$$c = [c_{-(n-1)}, \ldots, c_0, \ldots, c_{n-1}]^T$$

to be a column vector formed from the entries of T. Let

$$d = [d_0, \ldots, d_{2n-2}]^T$$

be an unknown column vector. Since V is nonsingular, we may choose d such that

$$c = Vd.$$

That is,

$$(1) \qquad c_i = \sum_{t=0}^{n-1} d_t\epsilon_t{}^i + \sum_{t=n}^{2n-2} d_t\epsilon_t{}^i, \qquad -n < i < n.$$

Equations (1) are the same as the matrix equation

$$(2) \qquad T_{n-1} = FDF^* + G\Delta G^*,$$

where

$$D = \mathrm{diag}(d_0, \ldots, d_{n-1}),$$

$$\Delta = \mathrm{diag}(d_n, \ldots, d_{2n-2}),$$

$$F = [\epsilon_t{}^i]_{0\leqq i\leqq n-1,\ 0\leqq t\leqq n-1},$$

$$G = [\epsilon_t{}^i]_{0\leqq i\leqq n-1,\ n\leqq t\leqq 2n-2}.$$

Here F is square, G rectangular, and * denotes conjugate transpose. We shall deduce the Fischer-Frobenius theorem from (2). Note that both FDF^* and $G\Delta G^*$ are Toeplitz matrices.

3. The Fischer-Frobenius Theorem

Take a and b to be fixed complex numbers, with $a\bar{b}$ not real. Let

$$\xi = [\xi_0, \ldots, \xi_{n-1}]^T, \qquad \eta = [\eta_0, \ldots, \eta_{n-1}]^T$$

be column vectors, and λ an indeterminate. If the right-hand side of the polynomial identity

(3) $$\sum_{p=0}^{n-1} \xi_p \lambda^p = \sum_{j=0}^{n-1} (a + \bar{a}\lambda)^{n-1-j} (b + \bar{b}\lambda)^j \eta_j$$

is multiplied out, then compared with the left-hand side, each ξ_p is linearly expressed in the η_j, $j = 0, \ldots, n-1$, with coefficients depending on a and b. Thus we obtain a matrix A for which

(4) $$\xi = A\eta;$$

explicit formulas for the entries of A may be seen in [1, p. 209]. (We shall not need these formulas.) Let $A = [\alpha_{pj}]_{0 \le p, j \le n-1}$. This matrix A is nonsingular. Indeed, set $\xi = 0$ in (3) and (4), then replace λ in (3) with $(a\lambda - b)/(\bar{b} - \bar{a}\lambda)$ to obtain

$$\sum_{j=0}^{n-1} \lambda^j \eta_j = 0,$$

after cancelling a factor. Thus $\xi = 0$ implies $\eta = 0$; hence A is nonsingular.

In (3) and (4) set $\eta_0 = \ldots = \eta_{j-1} = \eta_{j+1} = \ldots = \eta_{n-1} = 0$, $\eta_j = 1$, for fixed j. Then, by (4), $\xi_p = \alpha_{pj}$, and hence by (3)

(5) $$\sum_{p=0}^{n-1} \alpha_{pj} \lambda^p = (a + \bar{a}\lambda)^{n-1} \left(\frac{b + \bar{b}\lambda}{a + \bar{a}\lambda}\right)^j.$$

The theorem of Fischer and Frobenius, slightly generalized is this:

THEOREM 1: *Let matrix* A *be defined as above. Then, for any Toeplitz matrix* T_{n-1}, *the matrix*

(6) $$\mathrm{H}_{n-1} = \mathrm{A}^{\mathrm{T}} \mathrm{T}_{n-1} \bar{\mathrm{A}}$$

is a Hankel matrix. Conversely, for any Hankel matrix H_{n-1}, *the matrix* T_{n-1} *defined by* (6) *is a Toeplitz matrix.*

PROOF: Choose $\epsilon_0, \ldots, \epsilon_{2n-2}$ on the unit circle, distinct and unequal to $-a/\bar{a}$. We then have the decomposition (2). Since Hankel matrices are closed under addition, it will suffice to prove that both $A^T FDF^* \bar{A}$ and $A^T G\Delta G^* \bar{A}$ are Hankel matrices. We give the proof for the first of these, the proof for the other being similar. Let $f(\lambda) = (b + \bar{b}\lambda)/(a + \bar{a}\lambda)$. This Möbius function f maps the unit circle to the extended real axis, and in particular maps $\epsilon_0, \ldots, \epsilon_{2n-2}$ to finite real values. Set

$$r_i = f(\epsilon_i), \qquad 0 \leqq i \leqq 2n-2,$$

$$\rho_i = (a + \bar{a}\epsilon_i)^{n-1}, \qquad 0 \leqq i \leqq 2n-2.$$

Then each r_i is real. The (j, l) element of $A^T F$ is (by (5))

$$\sum_{p=0}^{n-1} \alpha_{pj} \epsilon_l^p = \rho_l f(\epsilon_l)^j = \rho_l r_l^j.$$

Thus

$$A^T F = F_1 D_1$$

where

$$F_1 = [r_l^i]_{0 \leqq i \leqq n-1,\ 0 \leqq l \leqq n-1}$$

$$D_1 = \mathrm{diag}(\rho_0, \ldots, \rho_{n-1}).$$

Thus F_1 is a real Vandermonde matrix. We now have

$$A^T F D F^* \bar{A} = (A^T F) D (A^T F)^*$$

$$= F_1(D_1 D D_1^*) F_1^* = F_1 D_2 F_1^T,$$

where $D_2 = D_1 D D_1^*$ is diagonal. However, any matrix of the form $F_1 D_2 F_2^T$ with F_1 a Vandermonde matrix and D_2 diagonal is a Hankel matrix.

Thus the transformation $T_{n-1} \to A^T T_{n-1} \bar{A}$ carries Toeplitz matrices to Hankel matrices. This evidently is a nonsingular linear transformation from the $(2n-1)$-dimensional complex vector space of Toeplitz matrices into (and therefore onto) the $2n-1$ dimensional complex space of Hankel matrices. Thus any Hankel matrix H_{n-1} is (uniquely) realizable in the form $A^T T_{n-1} \bar{A}$. This completes the proof.

When T_{n-1} is Hermitian, H_{n-1} is also Hermitian, being a conjuctive transform of T_{n-1}. Conversely, when H_{n-1} is Hermitian, so is its conjunctive transform $T_{n-1} = A^{-1^T} H_{n-1} \bar{A}^{-1}$. Thus we have:

COROLLARY: *Under the Fischer-Frobenius transformation, Hermitian Toeplitz matrices map onto real Hankel matrices, and conversely.*

The preparation of this paper was supported in part by the Air Force Office of Scientific Research, under Grant 72–2164.

3. References

[1] Iohvidov, I. C., Toeplitz and Hankel matrices and forms, Moscow, 1974 (in Russian).

[2] Iohvidov, I. C., On the transformation of Fischer-Frobenius, Theory of functions, functional analysis and its applications, **15** (1972), 203–212 (in Russian).

[3] Iohvidov, I. C., and Krein, M. G., Spectral theory of operators in spaces with an indefinite metric II, American Math. Society Translations, Series 2, vol. **34,** pp. 283–373.

(Paper 80B2–442)