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Real, 3 × 3, D-Stable Matrices*

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We characterize the 3×3 real D-stable Matrices.

Key words: Arc-connectedness; Carlson-Johnson conjecture; D-stable; positive stable matrix.

We give a complete description of the 3×3, real, D-stable matrices and thereby extend one of the main theorems in [2].[1] One nice feature of our characterizing conditions is that the requisite invariance under action by the multiplicative group of diagonal matrices with positive diagonal entries and the requisite invariance under inversion are clear. Our result is more evidence that the set of D-stable matrices is complicated, and we hope that it will assist the search for a general description, by helping with the formulation of conjectures and by providing counterexamples. For more information on this extensively studied problem see [3].

We call the square matrix M (positive) *stable* provided that $\mathrm{Re}(\lambda) > 0$ for every eigenvalue λ of M. And M is *D-stable* provided that MD is stable for every diagonal matrix D whose diagonal entries are positive.

Let $M = (m_{ij})$ be a 3×3 matrix with real entries. The principal minors of M will be denoted: $a = m_{11}$, $b = m_{22}$, $c = m_{33}$, $A = m_{22}m_{33} - m_{32}m_{23}$, $B = m_{11}m_{33} - m_{31}m_{13}$, $C = m_{11}m_{22} - m_{21}m_{12}$, $\delta = \det(M)$. We shall say that a and A, b and B, c and C, and 1 and δ are supplementary principal minors of each other. We say that M is of type 1 if some principal minor of M vanishes without its supplement vanishing also. Otherwise M is of type 2.

Let $\Delta = \sqrt{aA} + \sqrt{bB} + \sqrt{cC}$.

THEOREM: *M is D-stable if and only if*

$$\begin{array}{ll} \text{(i)} & \text{a, b, c, A, B, C } \textit{are non-negative,} \\ \text{(ii)} & a+b+c,\ A+B+C,\ \delta \textit{ are positive, and} \\ \text{(iii)} & \delta \leq \Delta^2 \textit{ if } M \textit{ is of type 1;} \\ & \delta < \Delta^2 \textit{ if } M \textit{ is of type 2.} \end{array}$$

PROOF: Let $D = \mathrm{diag}(x, y, z)$. The Routh-Hurwitz Theorem [5], applied to the characteristic polynomial of MD (cf. [2]), says that MD is stable if and only if

$$\begin{array}{ll} (\alpha) & ax + by + cz > 0 \\ (\beta) & (ax+by+cz)(Ayz+Bxz+Cxy) - xyz\delta > 0 \\ (\gamma) & xyz\delta > 0. \end{array}$$

Thus M is D-stable if and only if (α), (β), (γ) hold for all $x, y, z > 0$. That (α) holds for all $x, y, z > 0$ is clearly equivalent to "$a, b, c \geq 0$ and $a+b+c > 0$". That (γ) holds is equivalent to "$\delta > 0$". When (α) and (γ) hold then (β) implies that $Ayz + Bxz + Cxy > 0$. If M is D-stable this is true for all

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[1] Figures in brackets indicate the literature references at the end of this paper.

$x, y, z > 0$, and so $A, B, C \geqslant 0$ and $A+B+C > 0$. To summarize: M is D-stable implies (i) and (ii), and (i), (ii) imply that (α), (γ) hold for all $x, y, z > 0$ (cf. [2]). We pause here to prove:

LEMMA: *Suppose that the principal minors of* M *satisfy (i) and (ii).*

Let $f(x, y, z) = (ax+by+cz)(Ayz+Bxz+Cxy)/xyz$ *and let* J *denote the infimum of* $f(x, y, z)$ *for* $x, y, z > 0$. *Then*

(1) $J = \Delta^2$

(2) $f(x, y, z) > \Delta^2$ *for all* $x, y, z > 0$ *if* M *is of type 1*

(3) $f(x, y, z) = \Delta^2$ *for some* $x, y, z > 0$ *if* M *is of type 2.*

PROOF: A bit of algebra shows:

$$(*)\quad xyz(f(x, y, z) - \Delta^2) = z(\sqrt{aB}\,x - \sqrt{bA}\,y)^2 + y(\sqrt{aC}\,x - \sqrt{cA}\,z)^2 + x(\sqrt{bC}\,y - \sqrt{cB}\,z)^2.$$

Thus $J \geqslant \Delta^2$, and when M is of type 2 we can verify (1) and (3) by selecting $x, y, z > 0$ so that the quadratic terms vanish. Indeed, if M is of type 2 then the coefficients of each quadratic term either both vanish or are both nonzero. Thus, a quadratic term which is not identically zero will be zero for appropriate positive values of its variables. If none of the quadratic terms is identically zero then all will vanish if $x = \sqrt{cA/aC}$, $y = \sqrt{cB/bC}$, $z = 1$. In all other cases the choice of $x, y, z > 0$ is easy because at least two of the quadratic terms must vanish identically.

For M's of type 1 we have yet to prove (1) and (2). In this case at least one nonzero principal minor has a vanishing supplement and we change the notation if necessary so that $cC = 0$, $c+C > 0$. We define T so that equation $(*)$ divided by xyz is $f(x, y, z) - \Delta^2 = [(\sqrt{aB}\,x - \sqrt{bA}\,y)^2/xy] + T$. If $c = 0$ and $C > 0$ then $T = C(ax+by)/z$, and if $c > 0$ and $C = 0$ then $T = cz(A/x + B/y)$. Thus in both cases (i) and (ii) imply that $T > 0$ for all $x, y, z > 0$ and (2) follows. Furthermore, in both cases, if we fix $x, y > 0$ we can manipulate $z > 0$ to make T small. Thus we can establish (1) by finding $x, y > 0$ for which the quadratic term divided by xy is small. If $aB = bA = 0$ this is trivial. If $aBbA \neq 0$ we set $x = \sqrt{bA}$, $y = \sqrt{aB}$ so that the quadratic term vanishes. In the only other case exactly one of the coefficients $\sqrt{aB}$, $\sqrt{bA}$ is nonzero. We then set the variable it multiplies equal to 1 and makes the other of the two variables x, y very large.

We now return to the proof of the theorem. If M is D-stable then equation (β) holds for all $x, y, z > 0$. By the lemma that implies that $\Delta^2 \geqslant \delta$ if M is of type 1 and $\Delta^2 > \delta$ if M is of type 2. In other words (iii) holds. Conversely, when (iii) holds the lemma shows that for all $x, y, z > 0$, $xyz\,\delta \leqslant zyx\,\Delta^2 < xyz\,f(x, y, z)$ if M is of type 1, $xyz\,\delta < xyz\,\Delta^2 \leqslant xyz\,f(x, y, z)$ if M is of type 2. Thus equation (β) holds for all $x, y, z > 0$ and M is D-stable.

COROLLARY: *The real 3×3 matrix* M *is* D*-stable if and only if* M+D *is* D*-stable for all diagonal* D *with nonnegative entries.*

PROOF: The converse is trivial, take $D = 0$.

Now assume that M is D-stable. Since $M+D$ may be viewed as the result of three successive pertubations of M by diagonal matrices each having at most one nonzero entry, there is no loss in assuming D has only one nonzero entry. Moreover, we may assume that $D = \text{diag}(x, 0, 0)$ with $x > 0$. Since $M+D = x\left[\frac{1}{x}M + \text{diag}(1, 0, 0)\right]$ and $\frac{1}{x}M$ is D-stable, it will be sufficient if we can prove that $M' = M + \text{diag}(1, 0, 0)$ is D-stable if M is. Let $a', b', \ldots, C', \delta', \Delta'$ be defined with respect to M' just as $a, b, \ldots, \Delta$ were with respect to M. Then $a' = a+1$, $b' = b$, $c' = c$, $A' = A$, $B' = B + c$, $C' = C + b$, $\delta' = \delta + A$. Thus M' has properties (i) and (ii). Since $(\Delta')^2 - \delta' \geqslant \Delta^2 - \delta + 2bc$ and $2bc$,

$\Delta^2 - \delta$ are nonnegative, we are sure that M' is D-stable except in the case (which we will show never occurs) that $\Delta^2 - \delta = bc = 0$ and M' is of type 2. In that case a principal minor of M' vanishes if and only if its supplement does. Then $B' = B + c > 0 \Rightarrow b' = b > 0 \Rightarrow C' = C + b > 0 \Rightarrow c' = c > 0$. So the assumption $B' > 0$ contradicts $bc = 0$. Similarly the assumption $C' > 0$ also contradicts $bc = 0$. Thus $B' = C' = 0$, that is $b = c = B = C = 0$. Conditions (i) and (ii) now imply that a, A are positive. This means that M is of type 2 and since M is D-stable $\Delta^2 - \delta > 0$, a contradiction.

The preceding corollary is more evidence for the Carlson-Johnson Conjecture [1] that if M is $n \times n$ and D-stable then $M + D$ is stable for all diagonal D with positive diagonal entries.

COROLLARY: *The set of 3×3 real* D-*stable matrices is arc-connected.*

PROOF: If M is D-stable, then $(1-t)M + tI$ for $0 \leq t \leq 1$ is, by the preceding corollary, an arc of D-stable matrices connecting M to I.

Both corollaries hold in the 1×1 and 2×2 cases also.

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