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Nonnegative Sums of Roots of Unity

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Let q, n be integers > 1, and let $\rho_1, \rho_2, \ldots, \rho_n$ be distinct qth roots of unity. It is shown that $\rho_1^k + \rho_2^k + \ldots + \rho_n^k \geqslant 0$ for all integral k if and only if n is a divisor of q and the set $\{\rho_1, \rho_2, \ldots, \rho_n\}$ coincides with the set $\{1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}\}$, where $\zeta_n = \exp(2\pi i/n)$.

Key words. Algebraic numbers; conjugates, roots of unity.

The theorem proved in this note has its origins in problems concerning the characterization of the spectrum of a nonnegative matrix which have been studied by S. Friedland (as yet unpublished). The question answered by the theorem was posed to the author by Dr. Friedland.

THEOREM: *Let* q,n *be integers* > 1, *and let* $\rho_1, \rho_2, \ldots, \rho_n$ *be distinct* qth *roots of unity. Then* $\rho_1^k + \rho_2^k + \ldots + \rho_n^k \geqslant 0$ *for all integral* k *if and only if* n *is a divisor of* q *and the set* $\{\rho_1, \rho_2, \ldots, \rho_n\}$ *coincides with the set*

$$\{1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}\}, \textit{ where } \zeta_n = \exp(2\pi \mathrm{i}/\mathrm{n}).$$

PROOF. We first assume the nonnegativity of the sums. Then after a suitable renumbering we may put $\rho_r = \zeta^{a_r}$, where $\zeta = \zeta_q = \exp(2\pi i/q)$ is a primitive qth root of unity, and

$$0 \leqslant a_1 < a_2 < \ldots < a_n \leqslant q-1. \tag{1}$$

Put

$$\alpha_k = \zeta^{a_1 k} + \zeta^{a_2 k} + \ldots + \zeta^{a_n k}.$$

Then by the hypotheses of the theorem,

$$\alpha_k \geqslant 0 \qquad \text{for all integral } k. \tag{2}$$

It is of course sufficient to consider only those k such that $0 \leqslant k \leqslant q-1$, since $\alpha_{k+q} = \alpha_k$.

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We have

$$\sum_{k=0}^{q-1} \alpha_k = \sum_{r=1}^{n} \sum_{k=0}^{q-1} \zeta^{a_r k} = \begin{cases} 0, & a_1 \neq 0 \\ q, & a_1 = 0. \end{cases}$$

Thus if $a_1 \neq 0$, $\alpha_0 + \alpha_1 + \ldots + \alpha_{q-1} = 0$, which implies by (2) that $\alpha_0 = \alpha_1 = \ldots = \alpha_{q-1} = 0$. This is not possible, since $\alpha_0 = n$. It follows that $a_1 = 0$, and that

$$\alpha_0 + \alpha_1 + \ldots + \alpha_{q-1} = q. \tag{3}$$

The proof will be by induction on $\Omega(q)$, the total number of prime factors of q. Consider first $\Omega(q) = 1$, so that q is prime. Since q is prime, the algebraic integers $\alpha_1, \alpha_2, \ldots, \alpha_{q-1}$ form a complete set of conjugates; and since an algebraic number is zero if and only if every one of its conjugates is zero, there are just two possibilities:

(a) $$\alpha_k \neq 0, \qquad 1 \leqslant k \leqslant q-1.$$

(b) $$\alpha_k = 0, \qquad 1 \leqslant k \leqslant q-1.$$

Assume first that (a) holds. Then (2) implies that $\alpha_k > 0$, $1 \leqslant k \leqslant q-1$. We have from (3) that

$$1 = (\alpha_0 + \alpha_1 + \ldots + \alpha_{q-1})/q \geqslant (\alpha_0 \alpha_1 \ldots \alpha_{q-1})^{1/q},$$

$$\alpha_0 \alpha_1 \ldots \alpha_{q-1} \leqslant 1,$$

$$\alpha_1 \alpha_2 \ldots \alpha_{q-1} < 1,$$

since $\alpha_0 = n < 1$. It follows that $0 < \alpha_1 \alpha_2 \ldots \alpha_{q-1} < 1$, which is a contradiction, since $\alpha_1 \alpha_2 \ldots \alpha_{q-1}$ is the norm of the algebraic integer α_1 and so a rational integer.

Now assume that (b) holds, and that $n \leqslant q-1$. Then we must have that

$$\alpha_k = 0, \qquad 1 \leqslant k \leqslant n.$$

Thus

$$\sum_{r=1}^{n} \zeta^{a_r k} = 0, \qquad 1 \leqslant k \leqslant n. \tag{4}$$

Because of (1), the matrix $S = (\zeta^{a_r k})$, $1 \leqslant r, k \leqslant n$, must be non-singular, which contradicts (4). Thus the assumption $n \leqslant q-1$ is not possible, and so $n = q$, and the integers a_r must satisfy $a_r = r-1$, $1 \leqslant r \leqslant n$. Thus the result follows in this case.

Now suppose the result proved for all q such that $\Omega(q) < N$, $N \geqslant 2$, and let q be any positive integer such that $\Omega(q) = N$. Let t be any integer such that $1 \leqslant t \leqslant n$. We have

$$\alpha_k = \sum_{r=1}^{n} \zeta^{a_r k},$$

$$\zeta^{-a_t k} \alpha_k = \sum_{r=1}^{n} \zeta^{(a_r - a_t) k},$$

$$\sum_{k=0}^{q-1} \zeta^{-a_t k} \alpha_k = \sum_{r=1}^{n} \sum_{k=0}^{q-1} \zeta^{(a_r - a_t) k} = q,$$

sine (1) implies that $a_r - a_t \equiv 0 \bmod q$ if and only if $r = t$.
Thus

$$\sum_{k=0}^{q-1} \zeta^{-a_t k} \alpha_k = \sum_{k=0}^{q-1} a_k = q,$$

$$\sum_{k=1}^{q-1} (1 - \zeta^{-a_t k})\ \alpha_k = 0. \tag{5}$$

Now use the fact that α_k is real, take the complex conjugate of both sides of (5), and add the resulting equations. The result is that

$$\sum_{k=0}^{q-1} (2 - \zeta^{a_t k} - \zeta^{-a_t k})\ \alpha_k = 0. \tag{6}$$

Since $2 - \zeta^{a_t k} - \zeta^{-a_t k} \geqslant 0$, and $\alpha_k \geqslant 0$ by assumption, we conclude from (6) that

$$(2 - \zeta^{a_t k} - \zeta^{-a_t k})\alpha_k = 0;$$

and since

$$2 - \zeta^{a_t k} - \zeta^{-a_t k} = -\zeta^{-a_t k}\ (1 - \zeta^{a_t k})^2,$$

we may also conclude that

$$(1 - \zeta^{a_t k})\alpha_k = 0, \qquad 1 \leqslant t \leqslant n, \qquad \text{all integral } k. \tag{7}$$

Now the set $\{k : 1 \leqslant k \leqslant q\}$ coincides with the set $\{ql/d\colon d|q,\ (l, d) = 1\}$, where the notation means that d runs over the positive divisors of q and l runs over the positive integers $\leqslant d$ and relatively prime to d.

Suppose first that there is no value of $d > 1$ such that d divides a_t for all t. Then (7) and the decomposition above easily imply that $\alpha_k = 0$ for $1 \leqslant k \leqslant q-1$ (here we must use the fact that the conjugates of ζ_d are precisely ζ_d^l, $(l, d) = 1$, so that $\alpha_{q/d}$ vanishes if and only if $\alpha_{ql/d}$ vanishes). Then it follows as before that $n = q$, and that $a_r = r-1$, $1 \leqslant r \leqslant n$. Next suppose that there is a $d > 1$ such that d divides a_t for all t. Put $a_t = db_t$, $1 \leqslant t \leqslant n$.

Then

$$\alpha_k = \sum_{r=1}^{n} \zeta_q^{a_r k} = \sum_{k=1}^{n} \zeta_{q/d}^{b_r k} \geqslant 0 \text{ for all integral } k,$$

and

$$0 = b_1 < b_2 < \ldots < b_n \leqslant q/d - 1.$$

Since $\Omega(q/d) < N$, the induction hypothesis implies that n divides q/d, and that

$$b_r = q(r-1)/nd, \qquad 1 \leqslant r \leqslant n.$$

It follows that n divides q and that

$$a_r = db_r = q(r-1)/nd, \qquad 1 \leq r \leq n.$$

Hence the result holds for any q such that $\Omega(q) = N$, and the proof (in this direction) is complete.

Finally, if n divides q and $a_r = q(r-1)/n$, $1 \leq r \leq n$, then

$$\alpha_k = \sum_{r=1}^{n} \zeta^{q(r-1)/n} = \sum_{r=1}^{n} \zeta_n^{(r-1)k} = \begin{cases} n, & n/k \\ 0, & \text{otherwise} \end{cases}$$

so that $\alpha_k \geqslant 0$ for all k.

This completes the proof of the theorem.

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