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# Complete Elliptic Integrals Resulting from Infinite Integrals of Bessel Functions. II *

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Infinite integrals involving products of Bessel and trigonometric or hyperbolic functions reducible to complete elliptic integrals are compiled. The table also contains certain types of infinite double integrals of modified Bessel functions. All the results are expressed in conveniently compact form suited for practical applications.

Key words: Bessel functions; complete elliptic integrals; hyperbolic functions; infinite integrals; modified Bessel functions; noise theory; signal statistics; trigonometric functions.

## 1. Introduction

This table is primarily intended for applied mathematicians, physicists and engineers faced with the problem of evaluating infinite integrals involving Bessel functions.

In a previous paper,[1] infinite integrals of Bessel functions and their products were compiled in conveniently closed form. In this paper, a table is prepared for those involving products of Bessel and trigonometric or hyperbolic functions. It also contains certain kinds of infinite double integrals of modified Bessel functions, which are often encountered especially in the fields of signal statistics and noise theory.

Throughout this paper, the parameters are usually positive real and notations occurring several times on a section are explained at the top of the section.

## 2. Integrands Involving Products of Bessel and Trigonometric Functions

2.1. $k_1^2=\frac{a+b}{2a}, \qquad k_1^2+k_1'^2=1$

$k_2^2=\frac{2a}{a+b}$

$$\int_0^\infty J_0(ax^2)\cos(bx^2)\,dx=\frac{1}{2\sqrt{\pi a}}[K(k_1)+K(k_1')], \qquad a>b;$$

$$=\frac{k_2}{2\sqrt{\pi a}}K(k_2), \qquad a<b. \tag{1}$$

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[1] Okui, S., Complete elliptic integrals resulting from infinite integrals of Bessel functions, J. Res. Nat. Bur. Stand. (U.S.), **78B** (Math. Sci.), No. 3, 113–135 (July–Sept. 1974).

$$\int_0^\infty J_1(ax^2)\cos(bx^2)\,dx = \frac{1}{2\sqrt{\pi a}}[2E(k_1) - K(k_1) + 2E(k_1') - K(k_1')], \qquad a > b;$$

$$= -\frac{1}{2k_2\sqrt{\pi a}}[(2-k_2^2)K(k_2) - 2E(k_2)], \qquad a < b. \tag{2}$$

$$\int_0^\infty J_1(ax^2)\cos(bx^2)\,x^{-2}dx = \frac{2\sqrt{a}}{3\sqrt{\pi}}[(1-k_1^2)K(k_1) - (1-2k_1^2)E(k_1)$$

$$+ (1-k_1'^2)K(k_1') - (1-2k_1'^2)E(k_1')], \qquad a > b;$$

$$= \frac{2\sqrt{a}}{3k_2^3\sqrt{\pi}}[(2-k_2^2)E(k_2) - 2(1-k_2^2)K(k_2)], \qquad a < b. \tag{3}$$

$$\int_0^\infty J_2(ax^2)\cos(bx^2)\,dx = -\frac{1}{6\sqrt{\pi a}}[8(1-2k_1^2)E(k_1) - (5-8k_1^2)K(k_1)$$

$$+ 8(1-2k_1'^2)E(k_1') - (5-8k_1'^2)K(k_1')], \qquad a > b;$$

$$= -\frac{1}{6k_2^3\sqrt{\pi a}}[(4-k_2^2)(4-3k_2^2)K(k_2) - 8(2-k_2^2)E(k_2)], \qquad a < b. \tag{4}$$

$$\int_0^\infty J_2(ax^2)\cos(bx^2)\,x^{-2}dx = \frac{2\sqrt{a}}{15\sqrt{\pi}}[(1-k_1^2)(1-8k_1^2)K(k_1) - (1-16k_1^2+16k_1^4)E(k_1)$$

$$+ (1-k_1'^2)(1-8k_1'^2)K(k_1') - (1-16k_1'^2+16k_1'^4)E(k_1')], \qquad a > b;$$

$$= -\frac{2\sqrt{a}}{15k_2^5\sqrt{\pi}}[(16-16k_2^2+k_2^4)E(k_2) - 8(1-k_2^2)(2-k_2^2)K(k_2)], \qquad a < b. \tag{5}$$

$$\int_0^\infty J_2(ax^2)\cos(bx^2)\,x^{-4}\,dx$$

$$= \frac{8\sqrt{a^3}}{105\sqrt{\pi}}[(1-k_1^2)(2+5k_1^2-8k_1^4)K(k_1) - 2(1-2k_1^2)(1+4k_1^2-4k_1^4)E(k_1)$$

$$+ (1-k_1'^2)(2+5k_1'^2-8k_1'^4)K(k_1') - 2(1-2k_1'^2)$$

$$\times (1+4k_1'^2-4k_1'^4)E(k_1')], \qquad a > b;$$

$$= \frac{8\sqrt{a^3}}{105k_2^7\sqrt{\pi}}[(1-k_2^2)(16-16k_2^2-k_2^4)K(k_2) - 2(2-k_2^2)$$

$$\times (4-4k_2^2-k_2^4)E(k_2)], \qquad a < b. \tag{6}$$

$$\int_0^\infty J_3(ax^2)\cos(bx^2)\,dx$$

$$= -\frac{1}{30\sqrt{\pi a}}[2(23-128k_1^2+128k_1^4)E(k_1) - (31-144k_1^2+128k_1^4)K(k_1)$$

$$+ 2(23 - 128k_1'^2 + 128k_1'^4)E(k_1') - (31 - 144k_1'^2$$

$$+ 128k_1'^4)K(k_1')], \qquad a > b;$$

$$= \frac{1}{30k_2^5\sqrt{\pi a}}[(2-k_2^2)(128 - 128k_2^2 + 15k_2^4)K(k_2) - 2(128$$

$$- 128k_2^2 + 23k_2^4)E(k_2)], \qquad a < b. \tag{7}$$

$$\int_0^\infty J_3(ax^2)\cos(bx^2)\,x^{-2}dx = -\frac{2\sqrt{a}}{105\sqrt{\pi}}[(1-k_1^2)(3 - 80k_1^2 + 128k_1^4)K(k_1) - (3 - 134k_1^2$$

$$+ 384k_1^4 - 256k_1^6)E(k_1) + (1 - k_1'^2)(3 - 80k_1'^2 + 128k_1'^4)K(k_1')$$

$$-(3 - 134k_1'^2 + 384k_1'^4 - 256k_1'^6)E(k_1')], \qquad a > b;$$

$$= -\frac{2\sqrt{a}}{105k_2^7\sqrt{\pi}}[(2-k_2^2)(128 - 128k_2^2 + 3k_2^4)E(k_2) - 2(1-k_2^2)$$

$$\times(128 - 128k_2^2 + 27k_2^4)K(k_2)], \qquad a < b. \tag{8}$$

$$\int_0^\infty J_3(ax^2)\cos(bx^2)\,x^{-4}dx = \frac{8\sqrt{a^3}}{945\sqrt{\pi}}[(1-k_1^2)(2 + 15k_1^2 - 144k_1^4 + 128k_1^6)K(k_1) - 2(1 + 7k_1^2$$

$$- 135k_1^4 + 256k_1^6 - 128k_1^8)E(k_1) + (1-k_1'^2)(2 + 15k_1'^2 - 144k_1'^4$$

$$+ 128k_1'^6)K(k_1') - 2(1 + 7k_1'^2 - 135k_1'^4 + 256k_1'^6 - 128k_1'^8)E(k_1')],$$

$$a > b;$$

$$= -\frac{8\sqrt{a^3}}{945k_2^9\sqrt{\pi}}[(1-k_2^2)(2-k_2^2)(128 - 128k_2^2 - k_2^4)K(k_2) - (256$$

$$- 512k_2^2 + 270k_2^4 - 14k_2^6 - 2k_2^8)E(k_2)], \qquad a < b. \tag{9}$$

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$$\int_0^\infty J_0(ax^2)\sin(bx^2)\,dx = \frac{1}{2\sqrt{\pi a}}[K(k_1) - K(k_1')], \qquad a > b;$$

$$= \frac{k_2}{2\sqrt{\pi a}}K(k_2), \qquad a < b. \tag{10}$$

$$\int_0^\infty J_0(ax^2)\sin(bx^2)\,x^{-2}\,dx = \frac{2\sqrt{a}}{\sqrt{\pi}}[E(k_1) - (1-k_1^2)K(k_1)$$

$$+ (1 - k_1'^2)K(k_1') - E(k_1')], \qquad a > b;$$

$$= \frac{2\sqrt{a}}{k_2\sqrt{\pi}} E(k_2), \qquad a < b. \tag{11}$$

$$\int_0^\infty J_1(ax^2)\sin(bx^2)\,dx = -\frac{1}{2\sqrt{\pi a}}[2E(k_1) - K(k_1) + K(k_1') - 2E(k_1')], \qquad a > b;$$

$$= \frac{1}{2k_2\sqrt{\pi a}}[(2 - k_2^2)K(k_2) - 2E(k_2)], \qquad a < b. \tag{12}$$

$$\int_0^\infty J_1(ax^2)\sin(bx^2)\,x^{-2}dx = \frac{2\sqrt{a}}{3\sqrt{\pi}}[(1 - k_1^2)K(k_1) - (1 - 2k_1^2)E(k_1)$$

$$+ (1 - 2k_1'^2)E(k_1') - (1 - k_1'^2)K(k_1')], \qquad a > b;$$

$$= \frac{2\sqrt{a}}{3k_2^3\sqrt{\pi}}[(2 - k_2^2)E(k_2) - 2(1 - k_2^2)K(k_2)], \qquad a < b. \tag{13}$$

$$\int_0^\infty J_1(ax^2)\sin(bx^2)\,x^{-4}dx = \frac{8\sqrt{a^3}}{15\sqrt{\pi}}[2(1 - k_1^2 + k_1^4)E(k_1) - (1 - k_1^2)(2 - k_1^2)K(k_1)$$

$$+ (1 - k_1'^2)(2 - k_1'^2)K(k_1') - 2(1 - k_1'^2 + k_1'^4)E(k_1')], \qquad a > b;$$

$$= \frac{8\sqrt{a^3}}{15k_2^5\sqrt{\pi}}[2(1 - k_2^2 + k_2^4)E(k_2) - (1 - k_2^2)(2 - k_2^2)K(k_2)], \qquad a < b. \tag{14}$$

$$\int_0^\infty J_2(ax^2)\sin(bx^2)\,dx = -\frac{1}{6\sqrt{\pi a}}[8(1 - 2k_1^2)E(k_1) - (5 - 8k_1^2)K(\mathrm{k}_1)$$

$$+ (5 - 8k_1'^2)K(k_1') - 8(1 - 2k_1'^2)E(k_1')], \qquad a > b;$$

$$= -\frac{1}{6k_2^3\sqrt{\pi a}}[(4 - k_2^2)(4 - 3k_2^2)K(k_2) - 8(2 - k_2^2)E(k_2)], \qquad a < b. \tag{15}$$

$$\int_0^\infty J_2(ax^2)\sin(bx^2)x^{-2}dx = -\frac{2\sqrt{a}}{15\sqrt{\pi}}[(1 - k_1^2)(1 - 8k_1^2)K(k_1) - (1 - 16k_1^2 + 16k_1^4)E(k_1)$$
$$+ (1 - 16k_1'^2 + 16k_1'^4)E(k_1')$$
$$- (1 - k_1'^2)(1 - 8k_1'^2)K(k_1')], \qquad a > b;$$

$$= \frac{2\sqrt{a}}{15k_2^5\sqrt{\pi}}[(16 - 16k_2^2 + k_2^4)E(k_2)$$
$$-8(1 - k_2^2)(2 - k_2^2)K(k_2)], \qquad a < b. \tag{16}$$

$$\int_0^\infty J_2(ax^2)\sin(bx^2)x^{-4}dx = \frac{8\sqrt{a^3}}{105\sqrt{\pi}}[(1 - k_1^2)(2 + 5k_1^2 - 8k_1^4)K(k_1)$$
$$-2(1 - 2k_1^2)(1 + 4k_1^2 - 4k_1^4)E(k_1) +$$

$$+2(1-2k_1'^2)(1+4k_1'^2-4k_1'^4)E(k_1')-(1-k_1'^2)$$
$$\times(2+5k_1'^2-8k_1'^4)K(k_1')], \qquad a>b;$$
$$=\frac{8\sqrt{a^3}}{105k_2^7\sqrt{\pi}}[(1-k_2^2)(16-16k_2^2-k_2^4)K(k_2)-2(2-k_2^2)$$
$$\times(4-4k_2^2-k_2^4)E(k_2)], \qquad a<b. \tag{17}$$

$$\int_0^\infty J_3(ax^2)\sin(bx^2)dx=\frac{1}{30\sqrt{\pi a}}[2(23-128k_1^2+128k_1^4)E(k_1)$$
$$-(31-144k_1^2+128k_1^4)K(k_1)$$
$$+(31-144k_1'^2+128k_1'^4)K(k_1')-2(23-128k_1'^2$$
$$+128k_1'^4)E(k_1')], \qquad a>b;$$

$$=-\frac{1}{30k_2^5\sqrt{\pi a}}[(2-k_2^2)(128-128k_2^2+15k_2^4)K(k_2)$$
$$-2(128-128k_2^2+23k_2^4)E(k_2)], \qquad a<b. \tag{18}$$

$$\int_0^\infty J_3(ax^2)\sin(bx^2)x^{-2}dx=-\frac{2\sqrt{a}}{105\sqrt{\pi}}[(1-k_1^2)(3-80k_1^2+128k_1^4)K(k_1)$$
$$-(3-134k_1^2+384k_1^4-256k_1^6)E(k_1)$$
$$+(3-134k_1'^2+384k_1'^4-256k_1'^6)E(k_1')$$
$$-(1-k_1'^2)(3-80k_1'^2+128k_1'^4)K(k_1')], \qquad a>b;$$

$$=-\frac{2\sqrt{a}}{105k_2^7\sqrt{\pi}}[(2-k_2^2)(128-128k_2^2+3k_2^4)E(k_2)$$
$$-2(1-k_2^2)(128-128k_2^2+27k_2^4)K(k_2)], \qquad a<b. \tag{19}$$

$$\int_0^\infty J_3(ax^2)\sin(bx^2)x^{-4}dx=-\frac{8\sqrt{a^3}}{945\sqrt{\pi}}[(1-k_1^2)(2+15k_1^2-144k_1^4+128k_1^6)K(k_1)$$
$$-2(1+7k_1^2-135k_1^4+256k_1^6-128k_1^8)E(k_1)$$
$$+2(1+7k_1'^2-135k_1'^4+256k_1'^6-128k_1'^8)E(k_1')$$
$$-(1-k_1'^2)(2+15k_1'^2-144k_1'^4+128k_1'^6)K(k_1')], \qquad a>b;$$

$$=\frac{8\sqrt{a^3}}{945k_2^9\sqrt{\pi}}[(1-k_2^2)(2-k_2^2)(128-128k_2^2-k_2^4)K(k_2)$$
$$-(256-512k_2^2+270k_2^4-14k_2^6-2k_2^8)E(k_2)], \qquad a<b. \tag{20}$$

**2.2.**[2] $k^2=\dfrac{2(\sqrt{b+a}-\sqrt{b-a})}{2\sqrt{b}+\sqrt{b+a}-\sqrt{b-a}}$, $\quad a<b$

$$\int_0^\infty J_0^2(ax^2)\cos(2bx^2)dx=\frac{k^2}{\sqrt{\pi^3}(\sqrt{b+a}-\sqrt{b-a})}[K(k)]^2. \tag{1}$$

[2] The results are also valid for the integrals involving $\sin(2bx^2)$ in place of $\cos(2bx^2)$.

$$\int_0^\infty J_1^2(ax^2)\cos(2bx^2)\,dx = -\frac{a^2k^2}{3\sqrt{\pi^3}\,(1-k^2)^2(\sqrt{b+a}-\sqrt{b-a})^5}$$
$$\times[(2-k^2)E(k)-2(1-k^2)K(k)]^2. \qquad (2)$$

$$\int_0^\infty J_2^2(ax^2)\cos(2bx^2)\,dx = \frac{a^4k^2}{105\sqrt{\pi^3}\,(1-k^2)^4(\sqrt{b+a}-\sqrt{b-a})^9}$$
$$\times[(1-k^2)(16-16k^2-k^4)K(k)-2(2-k^2)(4-4k^2-k^4)E(k)]^2. \qquad (3)$$

## 2.3. $k^2 = \frac{4ab}{(a+b)^2}$

$$\int_0^\infty J_1^2(ax)\sin(2bx)x^{-2}dx = b - \frac{2(a+b)}{3\pi a^2}[(a^2+b^2)E(k)-(a-b)^2K(k)]. \qquad (1)$$

$$\int_0^\infty J_2^2(ax)\sin(2bx)x^{-2}dx = \frac{b}{2} - \frac{2(a+b)}{15\pi a^4}[(a-b)^2(4b^2-a^2)K(k)+(a^4+9a^2b^2-4b^4)E(k)]. \qquad (2)$$

$$\int_0^\infty J_2(ax)J_1(ax)\sin(2bx)x^{-1}dx = \frac{b}{a} + \frac{1}{3\pi a^3}[(b-a)(4b^2-a^2)K(k)-(a+b)(a^2+4b^2)E(k)]. \qquad (3)$$

$$\int_0^\infty J_3(ax)J_1(ax)\sin(2bx)\,dx$$
$$= \frac{4b}{a^2} - \frac{1}{3\pi a^4(a+b)}[(4a^4-17a^2b^2+16b^4)K(k)-(7a^2+16b^2)(a+b)^2E(k)]. \qquad (4)$$

$$\int_0^\infty J_3(ax)J_2(ax)\sin(2bx)x^{-1}dx$$
$$= \frac{b}{a} + \frac{1}{15\pi a^5}[(b-a)(3a^4-20a^2b^2+32b^4)K(k)+(a+b)(3a^4+52a^2b^2-32b^4)E(k)]. \qquad (5)$$

## 2.4.[3] $k_1^2 = \frac{4ab}{(a+b)^2-c^2}, \quad k_2^2 = \frac{(a+b)^2-c^2}{4ab}$

**$a > b$**

$$\int_0^\infty J_0(ax)J_0(bx)\cos(cx)\,dx = \frac{k_1}{\pi\sqrt{ab}}K(k_1), \qquad 0<c<a-b;$$

$$= \frac{1}{\pi\sqrt{ab}}K(k_2), \qquad a-b<c<a+b.$$

OF 64 (1)

[3] For $c > a+b$ the integrals vanish.

$$\int_0^\infty J_1(ax)J_1(bx)\cos(cx)dx=\frac{1}{\pi k_1\sqrt{ab}}[(2-k_1^2)K(k_1)-2E(k_1)], \qquad 0<c<a-b;$$

$$=\frac{1}{\pi\sqrt{ab}}[K(k_2)-2E(k_2)], \qquad a-b<c<a+b. \qquad (2)$$

$$\int_0^\infty J_2(ax)J_2(bx)\cos(cx)dx=\frac{1}{3\pi k_1^3\sqrt{ab}}[(16-16k_1^2+3k_1^4)K(k_1)-8(2-k_1^2)E(k_1)],$$

$$0<c<a-b;$$

$$=\frac{1}{3\pi\sqrt{ab}}[8(1-2k_2^2)E(k_2)-(5-8k_2^2)K(k_2)],$$

$$a-b<c<a+b. \qquad (3)$$

$$\int_0^\infty J_3(ax)J_3(bx)\cos(cx)dx=\frac{1}{15\pi k_1^5\sqrt{ab}}[(128-128k_1^2+15k_1^4)(2-k_1^2)K(k_1)$$

$$-2(128-128k_1^2+23k_1^4)E(k_1)], \qquad 0<c<a-b;$$

$$=\frac{1}{15\pi\sqrt{ab}}[(31-144k_2^2+128k_4^4)K(k_2)$$

$$-2(23-128k_2^2+128k_2^4)E(k_2)], \qquad a-b<c<a+b. \qquad (4)$$

**2.5.**[4] $k_1^2=\frac{c^2-(a-b)^2}{4ab}, \qquad k_2^2=\frac{4ab}{c^2-(a-b)^2}$

**$a>b$**

$$\int_0^\infty J_0(ax)J_0(bx)\sin(cx)dx=\frac{1}{\pi\sqrt{ab}}K(k_1), \qquad a-b<c<a+b;$$

$$=\frac{k_2}{\pi\sqrt{ab}}K(k_2), \qquad c>a+b. \qquad \text{OF 169} \qquad (1)$$

$$\int_0^\infty J_1(ax)J_1(bx)\sin(cx)dx=-\frac{1}{\pi\sqrt{ab}}[K(k_1)-2E(k_1)], \qquad a-b<c<a+b;$$

$$=-\frac{1}{\pi k_2\sqrt{ab}}[(2-k_2^2)K(k_2)-2E(k_2)], \qquad c>a+b. \qquad (2)$$

$$\int_0^\infty J_2(ax)J_2(bx)\sin(cx)dx=\frac{1}{3\pi\sqrt{ab}}[8(1-2k_1^2)E(k_1)$$

$$-(5-8k_1^2)K(k_1)], \qquad a-b<c<a+b.$$

$$=\frac{1}{3\pi k_2^3\sqrt{ab}}[(16-16k_2^2+3k_2^4)K(k_2)-$$

[4] For $0<c<a-b$ the integrals vanish.

$$-8(2-k_2^2)E(k_2)], \qquad c > a+b. \tag{3}$$

$$\int_0^\infty J_3(ax)J_3(bx)\sin(cx)\,dx = -\frac{1}{15\pi\sqrt{ab}}\,[(31-144k_1^2+128k_1^4)K(k_1)$$

$$-2(23-128k_1^2+128k_1^4)E(k_1)], \qquad a-b<c<a+b;$$

$$= -\frac{1}{15\pi k_2^5\sqrt{ab}}\,[(128-128k_2^2+15k_2^4)(2-k_2^2)K(k_2)$$

$$-2(128-128k_2^2+23k_2^4)E(k_2)], \qquad c > a+b. \tag{4}$$

**2.6.** $k_1^2 = \dfrac{4ab}{(a+b)^2-c^2}, \qquad k_2^2 = \dfrac{(a+b)^2-c^2}{4ab}$

$k_3^2 = \dfrac{c^2-(a+b)^2}{c^2-(a-b)^2}, \qquad a > b$

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$$\int_0^\infty Y_0(ax)Y_0(bx)\cos(cx)\,dx = \frac{k_1}{\pi\sqrt{ab}}K(k_1), \qquad 0<c<a-b;$$

$$= \frac{1}{\pi\sqrt{ab}}K(k_2), \qquad a-b<c<a+b;$$

$$= \frac{2\sqrt{1-k_3^2}}{\pi\sqrt{ab}}K(k_3), \qquad c > a+b. \qquad \text{OF 70} \tag{1}$$

**2.7.**[5] $k_1^2 = \dfrac{c^2-(a-b)^2}{4ab}, \qquad k_2^2 = \dfrac{4ab}{c^2-(a-b)^2}$

$a > b$

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$$\int_0^\infty Y_0(ax)J_0(bx)\cos(cx)\,dx = -\frac{1}{\pi\sqrt{ab}}K(k_1), \qquad a-b<c<a+b;$$

$$= -\frac{k_2}{\pi\sqrt{ab}}K(k_2), \qquad c > a+b. \qquad \text{OF 68} \tag{1}$$

$$\int_0^\infty Y_1(ax)J_1(bx)\cos(cx)\,dx = \frac{1}{\pi\sqrt{ab}}\,[K(k_1)-2E(k_1)], \qquad a-b<c<a+b;$$

$$= \frac{1}{\pi k_2\sqrt{ab}}\,[(2-k_2^2)K(k_2)-2E(k_2)], \qquad c > a+b. \tag{2}$$

$$\int_0^\infty Y_2(ax)J_2(bx)\cos(cx)\,dx = -\frac{1}{3\pi\sqrt{ab}}\,[8(1-2k_1^2)E(k_1)-(5-8k_1^2)K(k_1)],$$

$$a-b<c<a+b;$$

[5] For $0 < c < a-b$ the integrals vanish.

$$= -\frac{1}{3\pi k_2^3 \sqrt{ab}} [(16 - 16k_2^2 + 3k_2^4) K(k_2) - 8(2 - k_2^2)E(k_2)],$$

$$c > a + b. \quad (3)$$

$$\int_0^\infty Y_3(ax)J_3(bx)\cos(cx)dx = \frac{1}{15\pi\sqrt{ab}} [(31 - 144k_1^2 + 128k_1^4)K(k_1)$$

$$-2(23 - 128k_1^2 + 128k_1^4)E(k_1)], \quad a - b < c < a + b;$$

$$= \frac{1}{15\pi k_2^5 \sqrt{ab}} [(2 - k_2^2)(128 - 128k_2^2 + 15k_2^4)K(k_2)$$

$$-2(128 - 128k_2^2 + 23k_2^4)E(k_2)], \quad c > a + b. \quad (4)$$

**2.8.** $k_1^2 = \frac{(a-b)^2 - c^2}{(a+b)^2 - c^2}, \quad k_2^2 = \frac{c^2 - (a-b)^2}{4ab}$

$k_3^2 = \frac{4ab}{c^2 - (a-b)^2}, \quad a > b$

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$$\int_0^\infty J_0(ax)Y_0(bx)\cos(cx)\,dx = -\frac{2\sqrt{1-k_1^2}}{\pi\sqrt{ab}} K(k_1), \quad 0 < c < a - b;$$

$$= -\frac{1}{\pi\sqrt{ab}} K(k_2), \quad a - b < c < a + b;$$

$$= -\frac{k_3}{\pi\sqrt{ab}} K(k_3), \quad c > a + b. \qquad \text{OF 68} \quad (1)$$

$$\int_0^\infty J_1(ax)Y_1(bx)\cos(cx)\,dx = -\frac{2}{\pi\sqrt{ab(1-k_1^2)}} [2E(k_1) - (1-k_1^2)K(k_1)], \quad 0 < c < a - b;$$

$$= \frac{1}{\pi\sqrt{ab}} [K(k_2) - 2E(k_2)], \quad a - b < c < a + b;$$

$$= \frac{1}{\pi k_3 \sqrt{ab}} [(2 - k_3^2)K(k_3) - 2E(k_3)], \quad c > a + b. \quad (2)$$

$$\int_0^\infty J_2(ax)Y_2(bx)\cos(cx)dx = -\frac{2}{3\pi\sqrt{ab(1-k_1^2)^3}} [8(1+k_1^2)E(k_1)$$

$$-(1-k_1^2)(5+3k_1^2)K(k_1)], \quad 0 < c < a - b;$$

$$= -\frac{1}{3\pi\sqrt{ab}} [8(1-2k_2^2)E(k_2) - (5 - 8k_2^2)K(k_2)], \quad a - b < c < a + b;$$

$$= -\frac{1}{3\pi k_3^3 \sqrt{ab}} [(16 - 16k_3^2 + 3k_3^4)K(k_3)$$

$$-8(2 - k_3^2)E(k_3)], \quad c > a + b. \quad (3)$$



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$$\int_0^\infty J_3(ax)Y_3(bx)\cos(cx)dx = -\frac{2}{15\pi\sqrt{ab(1-k_1^2)^5}}[2(23+82k_1^2+23k_1^4)E(k_1)$$

$$-(1-k_1^2)(31+82k_1^2+15k_1^4)K(k_1)], \qquad 0<c<a-b;$$

$$=\frac{1}{15\pi\sqrt{ab}}[(31-144k_2^2+128k_2^4)K(k_2)$$

$$-2(23-128k_2^2+128k_2^4)E(k_2)], \qquad a-b<c<a+b;$$

$$=\frac{1}{15\pi k_3^5\sqrt{ab}}[(128-128k_3^2+15k_3^4)(2-k_3^2)K(k_3)$$

$$-2(128-128k_3^2+23k_3^4)E(k_3)], \qquad c>a+b. \qquad (4)$$

### 2.9. $k^2 = 1 - \frac{a^2}{b^2}$

$$\int_0^\infty Y_0(ax)J_0(ax)\sin(2bx)dx = -\frac{1}{\pi b}K(k), \qquad a<b;$$

$$=0, \qquad a>b. \qquad \text{OF 172} \qquad (1)$$

### 2.10. $k^2 = \frac{b^2}{a^2}, \quad a>b$

$$\int_0^\infty [Y_0(ax)J_1(ax)+J_0(ax)Y_1(ax)]\sin(2bx)dx = -\frac{2b}{\pi a^2}K(k). \qquad (1)$$

### 2.11. $k^2 = \frac{c^2-(a+b)^2}{c^2-(a-b)^2}$

$$\int_0^\infty [Y_0(ax)J_0(bx)+J_0(ax)Y_0(bx)]\sin(cx)dx = -\frac{2\sqrt{1-k^2}}{\pi\sqrt{ab}}K(k), \qquad c>a+b;$$

$$=0, \qquad 0<c<a+b. \qquad \text{OF 172} \qquad (1)$$

### 2.12. $k^2 = \frac{\sqrt{a^2+b^2}+b}{2\sqrt{a^2+b^2}}, \quad k^2+k'^2=1$

$$\int_0^\infty K_0(ax^2)\cos(bx^2)dx = \frac{\sqrt{\pi kk'}}{2\sqrt{a}}[K(k)+K(k')]. \qquad (1)$$

$$\int_0^\infty K_0(ax^2)\cos(bx^2)x^2dx = \frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^3}}[2E(k)-K(k)+2E(k')-K(k')]. \qquad (2)$$

$$\int_0^{\infty} K_0(ax^2)\cos(bx^2)x^4dx = -\frac{\sqrt{\pi(kk')^5}}{2\sqrt{a^5}}[8(1-2k^2)E(k)-(5-8k^2)K(k)$$

$$+8(1-2k'^2)E(k')-(5-8k'^2)K(k')]. \quad (3)$$

$$\int_0^{\infty} K_0(ax^2)\cos(bx^2)x^6dx = -\frac{\sqrt{\pi(kk')^7}}{2\sqrt{a^7}}[2(23-128k^2+128k^4)E(k)$$

$$-(31-144k^2+128k^4)K(k)+2(23-128k'^2$$

$$+128k'^4)E(k')-(31-144k'^2+128k'^4)K(k')]. \quad (4)$$

$$\int_0^{\infty} K_1(ax^2)\cos(bx^2)x^2dx = \frac{\sqrt{\pi kk'}}{2\sqrt{a^3}}[(1-k^2)K(k)-(1-2k^2)E(k)$$

$$+(1-k'^2)K(k')-(1-2k'^2)E(k')]. \quad (5)$$

$$\int_0^{\infty} K_1(ax^2)\cos(bx^2)x^4dx = \frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^5}}[(1-k^2)(1-8k^2)K(k)-(1-16k^2+16k^4)E(k)$$

$$+(1-k'^2)(1-8k'^2)K(k')-(1-16k'^2+16k'^4)E(k')]. \quad (6)$$

$$\int_0^{\infty} K_1(ax^2)\cos(bx^2)x^6dx = -\frac{\sqrt{\pi(kk')^5}}{2\sqrt{a^7}}[(1-k^2)(3-80k^2+128k^4)K(k)$$

$$-(3-134k^2+384k^4-256k^6)E(k)+(1-k'^2)(3-80k'^2+128k'^4)K(k')$$

$$-(3-134k'^2+384k'^4-256k'^6)E(k')]. \quad (7)$$

$$\int_0^{\infty} K_2(ax^2)\cos(bx^2)x^4dx = \frac{\sqrt{\pi kk'}}{2\sqrt{a^5}}[(1-k^2)(2+5k^2-8k^4)K(k)$$

$$-2(1-2k^2)(1+4k^2-4k^4)E(k)+(1-k'^2)(2+5k'^2-8k'^4)K(k')$$

$$-2(1-2k'^2)(1+4k'^2-4k'^4)E(k')]. \quad (8)$$

$$\int_0^{\infty} K_2(ax^2)\cos(bx^2)x^6dx = \frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^7}}[(1-k^2)(2+15k^2-144k^4$$

$$+128k^6)K(k)-2(1+7k^2-135k^4+256k^6-128k^8)E(k)$$

$$+(1-k'^2)(2+15k'^2-144k'^4+128k'^6)K(k')-2(1+7k'^2$$

$$-135k'^4+256k'^6-128k'^8)E(k')]. \quad (9)$$

---

$$\int_0^{\infty} K_0(ax^2)\sin(bx^2)dx = \frac{\sqrt{\pi kk'}}{2\sqrt{a}}[K(k)-K(k')]. \quad (10)$$

$$\int_0^{\infty} K_0(ax^2)\sin(bx^2)x^2dx = -\frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^3}}[2E(k)-K(k)+K(k')-2E(k')]. \quad (11)$$

$$\int_0^{\infty} K_0(ax^2)\sin(bx^2)x^4dx = -\frac{\sqrt{\pi(kk')^5}}{2\sqrt{a^5}}[8(1-2k^2)E(k)-(5-8k^2)K(k)$$

$$+(5-8k'^2)K(k')-8(1-2k'^2)E(k')]. \quad (12)$$

$$\int_0^\infty K_0(ax^2)\sin(bx^2)x^6dx = \frac{\sqrt{\pi}(kk')^7}{2\sqrt{a^7}}[2(23-128k^2+128k^4)E(k)$$
$$-(31-144k^2+128k^4)K(k)+(31-144k'^2+128k'^4)K(k')$$
$$-2(23-128k'^2+128k'^4)E(k')]. \tag{13}$$

$$\int_0^\infty K_0(ax^2)\sin(bx^2)x^{-2}dx = -\frac{\sqrt{\pi a}}{2\sqrt{kk'}}[2E(k)-K(k)+K(k')-2E(k')]. \tag{14}$$

$$\int_0^\infty K_1(ax^2)\sin(bx^2)dx = \frac{\sqrt{\pi}}{2\sqrt{akk'}}[E(k)-(1-k^2)K(k)+(1-k'^2)K(k')-E(k')]. \tag{15}$$

$$\int_0^\infty K_1(ax^2)\sin(bx^2)x^2dx = \frac{\sqrt{\pi kk'}}{2\sqrt{a^3}}[(1-k^2)K(k)-(1-2k^2)E(k)$$
$$+(1-2k'^2)E(k')-(1-k'^2)K(k')]. \tag{16}$$

$$\int_0^\infty K_1(ax^2)\sin(bx^2)x^4dx = -\frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^5}}[(1-k^2)(1-8k^2)K(k)-(1-16k^2+16k^4)E(k)$$
$$+(1-16k'^2+16k'^4)E(k')-(1-k'^2)(1-8k'^2)K(k')]. \tag{17}$$

$$\int_0^\infty K_1(ax^2)\sin(bx^2)x^6dx = -\frac{\sqrt{\pi(kk')^5}}{2\sqrt{a^7}}[(1-k^2)(3-80k^2+128k^4)K(k)$$
$$-(3-134k^2+384k^4-256k^6)E(k)+(3-134k'^2+384k'^4-256k'^6)E(k')$$
$$-(1-k'^2)(3-80k'^2+128k'^4)K(k')]. \tag{18}$$

$$\int_0^\infty K_2(ax^2)\sin(bx^2)x^2dx = \frac{\sqrt{\pi}}{2\sqrt{a^3kk'}}[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)$$
$$+(1-k'^2)(2-k'^2)K(k')-2(1-k'^2+k'^4)E(k')]. \tag{19}$$

$$\int_0^\infty K_2(ax^2)\sin(bx^2)x^4dx = \frac{\sqrt{\pi kk'}}{2\sqrt{a^5}}[(1-k^2)(2+5k^2-8k^4)K(k)$$
$$-2(1-2k^2)(1+4k^2-4k^4)E(k)+2(1-2k'^2)(1+4k'^2-4k'^4)E(k')$$
$$-(1-k'^2)(2+5k'^2-8k'^4)K(k')]. \tag{20}$$

$$\int_0^\infty K_2(ax^2)\sin(bx^2)x^6dx = \frac{\sqrt{\pi(kk')^3}}{2\sqrt{a^7}}[2(1+7k^2-135k^4+256k^6-128k^8)E(k)$$
$$-(1-k^2)(2+15k^2-144k^4+128k^6)K(k)+(1-k'^2)(2+15k'^2-144k'^{14}+128k'^{16})K(k')$$
$$-2(1+7k'^2-135k'^4+256k'^6-128'^8)E(k').]. \tag{21}$$

## 2.13. $k^2 = 1/2\left(1-\frac{\sqrt{2b}}{\sqrt{b+\sqrt{a^2+b^2}}}\right), \quad k^2+k'^2=1$

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$$\int_0^\infty K_0^2(ax^2)\cos(2bx^2)dx = \frac{\sqrt{\pi}(1-2k^2)}{4\sqrt{b}}[K(k')+K(k)]^2. \tag{1}$$

$$\int_0^\infty K_1(ax^2)K_0(ax^2)\cos(2bx^2)x^2dx = \frac{(1-2k^2)\sqrt{\pi}}{16kk'\sqrt{b(a^2+b^2)}}[K(k')+K(k)][k^2K(k')$$
$$-(1-2k'^2)E(k')+k'^2K(k)-(1-2k^2)E(k)]. \quad (2)$$

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$$\int_0^\infty K_0^2(ax^2)\sin(2bx^2)dx = \frac{\sqrt{\pi}(1-2k^2)}{4\sqrt{b}}[K(k')-K(k)]^2. \quad (3)$$

$$\int_0^\infty K_1(ax^2)K_0(ax^2)\sin(2bx^2)x^2dx = \frac{(1-2k^2)\sqrt{\pi}}{16kk'\sqrt{b(a^2+b^2)}}[K(k')-K(k)][k^2K(k')$$
$$-(1-2k'^2)E(k')-k'^2K(k)+(1-2k^2)E(k)]. \quad (4)$$

## 2.14. $k^2 = 1/2\left(1-\frac{\sqrt{2b}}{\sqrt{b+\sqrt{a^2+b^2}}}\right), \qquad k^2+k'^2=1$

$$\int_0^\infty K_0(ax^2)I_0(ax^2)\cos(2bx^2)dx = \frac{1-2k^2}{2\sqrt{\pi b}}K(k)[K(k')+K(k)]. \quad (1)$$

$$\int_0^\infty K_0(ax^2)I_0(ax^2)\sin(2bx^2)dx = \frac{1-2k^2}{2\sqrt{\pi b}}K(k)[K(k')-K(k)]. \quad (2)$$

## 2.15. $k^2 = \frac{b^2}{a^2+b^2}$

$$\int_0^\infty K_1(ax)K_0(ax)\cos(2bx)xdx = \frac{\pi k}{4ab}E(k). \quad (1)$$

$$\int_0^\infty K_2(ax)K_1(ax)\cos(2bx)x^3dx = \frac{\pi k}{16ab^3}[(1-k^2)(3-4k^2)K(k)-(3-13k^2+8k^4)E(k)]. \quad (2)$$

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$$\int_0^\infty K_1(ax)K_0(ax)\sin(2bx)x^2dx = \frac{\pi k}{8ab^2}[(1-k^2)K(k)-(1-2k^2)E(k)]. \quad (3)$$

$$\int_0^\infty K_2(ax)K_1(ax)\sin(2bx)\,x^4dx = \frac{3\pi k}{32ab^4}$$
$$\times[(1-k^2)(2+5k^2-8k^4)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \quad (4)$$

## 2.16. $k^2 = \frac{(a-b)^2+c^2}{(a+b)^2+c^2}$

$$\int_0^\infty K_0(ax)K_0(bx)\cos(cx)dx = \frac{\pi\sqrt{1-k^2}}{2\sqrt{ab}}K(k). \qquad \text{OF 88} \quad (1)$$

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$$\int_0^\infty K_0(ax)K_0(bx)\sin(cx)xdx = \frac{\pi c\sqrt{(1-k^2)^3}}{8k^2\sqrt{(ab)^3}}[K(k)-E(k)]. \quad (2)$$

$$\int_0^\infty K_1(ax)K_1(bx)\sin(cx)x\,dx=\frac{\pi c\sqrt{1-k^2}}{8k^2\sqrt{(ab)^3}}\,[(1+k^2)E(k)-(1-k^2)K(k)]. \tag{3}$$

## 2.17. $k^2=\dfrac{a^2}{a^2+b^2}$

$$\int_0^\infty K_1(ax)I_2(ax)\cos(2bx)x^{-1}dx=\frac{1}{3k^3}\,[4(1-k^2)K(k)-(4-5k^2)E(k)]-\frac{\pi b}{2a}. \tag{1}$$

$$\int_0^\infty K_2(ax)I_3(ax)\cos(2bx)x^{-1}dx=\frac{1}{15k^5}\,[4(1-k^2)(8+k^2)K(k)-(32-12k^2-23k^4)E(k)]-\frac{\pi b}{2a}. \tag{2}$$

## 2.18. $k^2=\dfrac{4ab}{(a+b)^2+c^2}$, $\quad a>b$

$$\int_0^\infty K_0(ax)I_0(bx)\cos(cx)\,dx=\frac{k}{2\sqrt{ab}}\,K(k). \qquad \text{OF 87} \tag{1}$$

$$\int_0^\infty K_1(ax)I_0(bx)\cos(cx)x\,dx=\frac{k}{16(1-k^2)a\sqrt{(ab)^3}} \times[k^2(a^2-b^2-c^2)E(k)+4ab(1-k^2)K(k)]. \tag{2}$$

$$\int_0^\infty K_1(ax)I_1(bx)\cos(cx)\,dx=\frac{1}{2k\sqrt{ab}}\,[(2-k^2)K(k)-2E(k)]. \tag{3}$$

$$\int_0^\infty K_2(ax)I_2(bx)\cos(cx)\,dx=\frac{1}{6k^3\sqrt{ab}}\,[(4-k^2)(4-3k^2)K(k)-8(2-k^2)E(k)]. \tag{4}$$

$$\int_0^\infty K_3(ax)I_3(bx)\cos(cx)\,dx=\frac{1}{30k^5\sqrt{ab}} \times[(128-128k^2+15k^4)(2-k^2)K(k)-2(128-128k^2+23k^4)E(k)]. \tag{5}$$

$$\int_0^\infty K_4(ax)I_4(bx)\cos(cx)\,dx=\frac{1}{210k^7\sqrt{ab}} \times[(6144-12288k^2+8000k^4-1856k^6+105k^8)K(k)-32(2-k^2)(96-96k^2+11k^4)E(k)]. \tag{6}$$

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$$\int_0^\infty K_0(ax)I_0(bx)\sin(cx)\,x\,dx=\frac{ck^3}{8(1-k^2)\sqrt{(ab)^3}}\,E(k). \tag{7}$$

$$\int_0^\infty K_1(ax)I_1(bx)\sin(cx)\,x\,dx=\frac{ck}{8(1-k^2)\sqrt{(ab)^3}}\,[(2-k^2)E(k)-2(1-k^2)K(k)]. \tag{8}$$

$$\int_0^\infty K_2(ax)I_2(bx)\sin(cx)\,x\,dx=\frac{c}{8k(1-k^2)\sqrt{(ab)^3}} \times[(16-16k^2+k^4)E(k)-8(1-k^2)(2-k^2)K(k)]. \tag{9}$$

$$\int_0^\infty K_3(ax)I_3(bx)\sin(cx)\,x\,dx=\frac{c}{24k^3(1-k^2)\sqrt{(ab)^3}}$$
$$\times[(2-k^2)(128-128k^2+3k^4)E(k)$$
$$-2(1-k^2)(128-128k^2+27k^4)K(k)]. \qquad (10)$$

**2.19.** $k^2=\dfrac{4ab}{(a+b)^2+c^2}=\sin^2\alpha, \qquad \sin\beta=\dfrac{c}{\sqrt{(a-b)^2+c^2}}$

$a>b$

**$\Lambda_0(\alpha,\beta)$ is Heuman's Lambda function**

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$$\int_0^\infty K_0(ax)I_1(bx)\cos(cx)x^{-1}dx=\frac{\sqrt{ab}}{bk}E(k)+\frac{k(b^2-a^2)}{4\sqrt{ab^3}}K(k)-\frac{\pi c}{4b}\Lambda_0(\alpha,\beta). \qquad (1)$$

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$$\int_0^\infty K_0(ax)I_1(bx)\sin(cx)dx=-\frac{ck}{4\sqrt{ab^3}}K(k)+\frac{\pi}{4b}\Lambda_0(\alpha,\beta). \qquad (2)$$

$$\int_0^\infty K_1(ax)I_2(bx)\sin(cx)dx=\frac{c}{k\sqrt{ab^3}}E(k)-\frac{ck(2a^2+b^2+c^2)}{4b\sqrt{(ab)^3}}K(k)+\frac{\pi a}{4b^2}\Lambda_0(\alpha,\beta)+\frac{\pi}{2a}. \qquad (3)$$

**2.20.** $k^2=1/2\left(1-\dfrac{a^2-b^2+c^2}{\sqrt{(a^2-b^2+c^2)^2+4a^2b^2}}\right), \qquad k^2+k'^2=1$

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$$\int_0^\infty K_0(ax)J_0(bx)\cos(cx)\,dx=\frac{\sqrt{kk'}}{\sqrt{ab}}K(k). \qquad (1)$$

$$\int_0^\infty K_1(ax)J_1(bx)\cos(cx)\,dx=\frac{1}{\sqrt{abkk'}}[E(k)-(1-k^2)K(k)]. \qquad (2)$$

$$\int_0^\infty K_2(ax)J_2(bx)\cos(cx)\,dx=\frac{1}{3\sqrt{ab(kk')^3}}$$
$$\times[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]. \qquad (3)$$

$$\int_0^\infty K_3(ax)J_3(bx)\cos(cx)\,dx=\frac{1}{15\sqrt{ab(kk')^5}}$$
$$\times[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]. \qquad (4)$$

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$$\int_0^\infty K_0(ax)J_0(bx)\sin(cx)\,x\,dx=\frac{c\sqrt{(kk')^3}}{\sqrt{(ab)^3}}[2E(k)-K(k)]. \qquad (5)$$

$$\int_0^\infty K_1(ax)J_1(bx)\sin(cx)x\,dx=\frac{c\sqrt{kk'}}{\sqrt{(ab)^3}}[(1-k^2)K(k)-(1-2k^2)E(k)]. \qquad (6)$$

$$\int_0^\infty K_2(ax)J_2(bx)\sin(cx)x\,dx=\frac{c}{\sqrt{(ab)^3}kk'}[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \quad (7)$$

$$\int_0^\infty K_3(ax)J_3(bx)\sin(cx)x\,dx=\frac{c}{3\sqrt{(abkk')^3}}[(8-15k^2+3k^4)(1-k^2)K(k)$$
$$-(8-19k^2+9k^4-6k^6)E(k)]. \quad (8)$$

## 3. Integrands Involving Products of Bessel and Hyperbolic Functions

### 3.1. $k^2=\frac{a+b}{2a}$, $k^2+k'^2=1$, $a>b$

$$\int_0^\infty K_0(ax^2)\cosh(bx^2)\,dx=\frac{\sqrt{\pi}}{2\sqrt{2a}}[K(k)+K(k')]. \quad (1)$$

$$\int_0^\infty K_0(ax^2)\cosh(bx^2)\,x^2dx=\frac{\sqrt{\pi}}{4(a^2-b^2)\sqrt{2a}}$$
$$\times[2aE(k)-(a-b)K(k)+2aE(k')-(a+b)K(k')]. \quad (2)$$

$$\int_0^\infty K_0(ax^2)\cosh(bx^2)\,x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a}}$$
$$\times[(a-b)(a-3b)K(k)+8abE(k)+(a+b)(a+3b)K(k')-8abE(k')]. \quad (3)$$

$$\int_0^\infty K_0(ax^2)\cosh(bx^2)\,x^6dx=\frac{\sqrt{\pi}}{16(a^2-b^2)^3\sqrt{2a}}$$
$$\times[2a(9a^2+23b^2)E(k)-(a-b)(9a^2-8ab+15b^2)K(k)$$
$$+2a(9a^2+23b^2)E(k')-(a+b)(9a^2+8ab+15b^2)K(k')]. \quad (4)$$

$$\int_0^\infty K_1(ax^2)\cosh(bx^2)\,x^2dx=\frac{\sqrt{\pi}}{4(a^2-b^2)\sqrt{2a}}$$
$$\times[(a-b)K(k)+2bE(k)+(a+b)K(k')-2bE(k')]. \quad (5)$$

$$\int_0^\infty K_1(ax^2)\cosh(bx^2)x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a}}[2(3a^2+b^2)E(k)$$
$$-(a-b)(3a-b)K(k)+2(3a^2+b^2)E(k')-(a+b)(3a+b)K(k')]. \quad (6)$$

$$\int_0^\infty K_1(ax^2)\cosh(bx^2)x^6dx=\frac{\sqrt{\pi}}{16(a^2-b^2)^3\sqrt{2a}}[(5a^2-24ab+3b^2)(a-b)K(k)$$
$$+2b(29a^2+3b^2)E(k)+(5a^2+24ab+3b^2)(a+b)K(k')-2b(29a^2+3b^2)E(k')]. \quad (7)$$

$$\int_0^\infty K_2(ax^2)\cosh(bx^2)x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a^3}}[(5a^2-3ab-4b^2)(a-b)K(k)$$
$$+8b(2a^2-b^2)E(k)+(a+b)(5a^2+3ab-4b^2)K(k')-8b(2a^2-b^2)E(k')]. \quad (8)$$

$$\int_0^\infty K_0(ax^2)\sinh(bx^2)dx=\frac{\sqrt{\pi}}{2\sqrt{2a}}[K(k)-K(k')]. \tag{9}$$

$$\int_0^\infty K_0(ax^2)\sinh(bx^2)x^2dx=\frac{\sqrt{\pi}}{4(a^2-b^2)\sqrt{2a}}[2aE(k)-(a-b)K(k)+(a+b)K(k')-2aE(k')]. \tag{10}$$

$$\int_0^\infty K_0(ax^2)\sinh(bx^2)x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a}}[(a-b)(a-3b)K(k)+8abE(k)+8abE(k')-(a+b)(a+3b)K(k')]. \tag{11}$$

$$\int_0^\infty K_0(ax^2)\sinh(bx^2)x^6dx=\frac{\sqrt{\pi}}{16(a^2-b^2)^3\sqrt{2a}}[2a(9a^2+23b^2)E(k)-(a-b)(9a^2-8ab+15b^2)K(k)+(a+b)(9a^2+8ab+15b^2)K(k')-2a(9a^2+23b^2)E(k')]. \tag{12}$$

$$\int_0^\infty K_1(ax^2)\sinh(bx^2)x^2dx=\frac{\sqrt{\pi}}{4(a^2-b^2)\sqrt{2a}}[(a-b)K(k)+2bE(k)+2bE(k')-(a+b)K(k')]. \tag{13}$$

$$\int_0^\infty K_1(ax^2)\sinh(bx^2)x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a}}[2(3a^2+b^2)E(k)-(a-b)(3a-b)K(k)+(a+b)(3a+b)K(k')-2(3a^2+b^2)E(k')]. \tag{14}$$

$$\int_0^\infty K_1(ax^2)\sinh(bx^2)x^6dx=\frac{\sqrt{\pi}}{16(a^2-b^2)^3\sqrt{2a}}[(5a^2-24ab+3b^2)(a-b)K(k)+2b(29a^2+3b^2)E(k)+2b(29a^2+3b^2)E(k')-(5a^2+24ab+3b^2)(a+b)K(k')]. \tag{15}$$

$$\int_0^\infty K_2(ax^2)\sinh(bx^2)x^4dx=\frac{\sqrt{\pi}}{8(a^2-b^2)^2\sqrt{2a^3}}[(5a^2-3ab-4b^2)(a-b)K(k)+8b(2a^2-b^2)E(k)+8b(2a^2-b^2)E(k')-(a+b)(5a^2+3ab-4b^2)K(k')]. \tag{16}$$

### 3.2. $k^2=\frac{b^2}{a^2}$, $a>b$

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$$\int_0^\infty K_1(ax)K_0(ax)\cosh(2bx)xdx=\frac{\pi}{4a^2(1-k^2)}E(k). \tag{1}$$

$$\int_0^\infty K_2(ax)K_1(ax)\cosh(2bx)x^3dx=\frac{\pi}{16a^4k^2(1-k^2)^3}[(3+7k^2-2k^4)E(k)-(1-k^2)(3+k^2)K(k)]. \tag{2}$$

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$$\int_0^\infty K_1(ax)K_0(ax)\sinh(2bx)x^2dx=\frac{\pi b}{8a^4k^2(1-k^2)^2}[(1+k^2)E(k)-(1-k^2)K(k)]. \tag{3}$$

$$\int_0^\infty K_2(ax)K_1(ax)\sinh(2bx)\,x^4dx=\frac{3\pi b}{32a^6k^4(1-k^2)^4}$$
$$\times[(1-k^2)(2-9k^2-k^4)K(k)-2(1+k^2)(1-6k^2+k^4)E(k)]. \quad (4)$$

**3.3.** $k_1^2=\dfrac{(a-b)^2-c^2}{(a+b)^2-c^2},\qquad k_2^2=\dfrac{c^2-(a-b)^2}{4ab}$

$a>b$

---

$$\int_0^\infty K_0(ax)K_0(bx)\cosh(cx)\,dx=\frac{\pi\sqrt{1-k_1^2}}{2\sqrt{ab}}K(k_1),\qquad c<a-b;$$
$$=\frac{\pi}{2\sqrt{ab}}K(k_2),\qquad a-b<c<a+b. \quad (1)$$

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$$\int_0^\infty K_0(ax)K_0(bx)\sinh(cx)\,x\,dx=\frac{\pi c\sqrt{(1-k_1^2)^3}}{8k_1^2\sqrt{(ab)^3}}[K(k_1)-E(k_1)],\qquad c<a-b;$$
$$=\frac{\pi c}{8k_2^2(1-k_2^2)\sqrt{(ab)^3}}[E(k_2)-(1-k_2^2)K(k_2)],$$
$$a-b<c<a+b. \quad (2)$$

$$\int_0^\infty K_1(ax)K_1(bx)\sinh(cx)x\,dx=\frac{\pi c\sqrt{1-k_1^2}}{8k_1^2\sqrt{(ab)^3}}[(1+k_1^2)E(k_1)-(1-k_1^2)K(k_1)],\qquad c<a-b;$$
$$=\frac{\pi c}{8k_2^2(1-k_2^2)\sqrt{(ab)^3}}[(1-k_2^2)K(k_2)-(1-2k_2^2)E(k_2)],$$
$$a-b<c<a+b. \quad (3)$$

**3.4.** $k^2=\dfrac{4ab}{(a+b)^2-c^2},\qquad a>b+c$

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$$\int_0^\infty K_0(ax)I_0(bx)\cosh(cx)\,dx=\frac{k}{2\sqrt{ab}}K(k). \quad (1)$$

$$\int_0^\infty K_1(ax)I_1(bx)\cosh(cx)\,dx=\frac{1}{2k\sqrt{ab}}[(2-k^2)K(k)-2E(k)]. \quad (2)$$

$$\int_0^\infty K_2(ax)I_2(bx)\cosh(cx)\,dx=\frac{1}{6k^3\sqrt{ab}}[(4-k^2)(4-3k^2)K(k)-8(2-k^2)E(k)]. \quad (3)$$

$$\int_0^\infty K_3(ax)I_3(bx)\cosh(cx)\,dx=\frac{1}{30k^5\sqrt{ab}}[(128-128k^2+15k^4)(2-k^2)K(k)$$
$$-2(128-128k^2+23k^4)E(k)]. \quad (4)$$

$$\int_0^\infty K_4(ax)I_4(bx)\cosh(cx)dx = \frac{1}{210k^7\sqrt{ab}}[(6144-12288k^2+8000k^4-1856k^6+105k^8)K(k)$$
$$-32(2-k^2)(96-96k^2+11k^4)E(k)]. \quad (5)$$

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$$\int_0^\infty K_0(ax)I_0(bx)\sinh(cx)xdx = \frac{ck^3}{8(1-k^2)\sqrt{(ab)^3}}E(k). \quad (6)$$

$$\int_0^\infty K_1(ax)I_1(bx)\sinh(cx)xdx = \frac{ck}{8(1-k^2)\sqrt{(ab)^3}}[(2-k^2)E(k)-2(1-k^2)K(k)]. \quad (7)$$

$$\int_0^\infty K_2(ax)I_2(bx)\sinh(cx)xdx = \frac{c}{8k(1-k^2)\sqrt{(ab)^3}}[(16-16k^2+k^4)E(k)$$
$$-8(1-k^2)(2-k^2)K(k)]. \quad (8)$$

$$\int_0^\infty K_3(ax)I_3(bx)\sinh(cx)xdx = \frac{c}{24k^3(1-k^2)\sqrt{(ab)^3}}[(2-k^2)(128-128k^2+3k^4)E(k)$$
$$-2(1-k^2)(128-128k^2+27k^4)K(k)]. \quad (9)$$

### 3.5. $k^2 = 1/2\left(1-\frac{a^2-b^2-c^2}{\sqrt{(a^2-b^2-c^2)^2+4a^2b^2}}\right), \quad k^2+k'^2=1$

**$a > c$**

---

$$\int_0^\infty K_0(ax)J_0(bx)\cosh(cx)dx = \frac{\sqrt{kk'}}{\sqrt{ab}}K(k). \qquad \text{ET II 15 (23)} \quad (1)$$

$$\int_0^\infty K_1(ax)J_1(bx)\cosh(cx)dx = \frac{1}{\sqrt{abkk'}}[E(k)-(1-k^2)K(k)]. \quad (2)$$

$$\int_0^\infty K_2(ax)J_2(bx)\cosh(cx)dx = \frac{1}{3\sqrt{ab(kk')^3}}$$
$$\times[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]. \quad (3)$$

$$\int_0^\infty K_3(ax)J_3(bx)\cosh(cx)dx = \frac{1}{15\sqrt{ab(kk')^5}}$$
$$\times[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]. \quad (4)$$

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$$\int_0^\infty K_0(ax)J_0(bx)\sinh(cx)xdx = \frac{c\sqrt{(kk')^3}}{\sqrt{(ab)^3}}[2E(k)-K(k)]. \quad (5)$$

$$\int_0^\infty K_1(ax)J_1(bx)\sinh(cx)xdx = \frac{c\sqrt{kk'}}{\sqrt{(ab)^3}}[(1-k^2)K(k)-(1-2k^2)E(k)]. \quad (6)$$

$$\int_0^\infty K_2(ax)J_2(bx)\sinh(cx)xdx = \frac{c}{\sqrt{(ab)^3kk'}}[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \quad (7)$$

$$\int_0^\infty K_3(ax)J_3(bx)\sinh(cx)x\,dx = \frac{c}{3\sqrt{(abkk')^3}}[(8-15k^2+3k^4)(1-k^2)K(k) - (8-19k^2+9k^4-6k^6)E(k)]. \quad (8)$$

**3.6.** $k^2 = 1/2\left(1 - \frac{a^2-b^2-c^2}{\sqrt{(a^2-b^2-c^2)^2+4a^2b^2}}\right)$,

$$cn^2u = \frac{2b^2}{\sqrt{(a^2-b^2-c^2)^2+4a^2b^2}-(a^2-b^2-c^2)},$$

**$cn(u)$, $sn(u)$, $dn(u)$ are Jacobian elliptic functions**

$a > c$

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$$\int_0^\infty K_1(ax)J_0(bx)\sinh(cx)\,dx = \frac{1}{a}[uE(k) - K(k)E(u) + \frac{sn(u)dn(u)}{cn(u)}K(k)]. \quad \text{ET II 15 (24)} \quad (1)$$

## 4. Miscellaneous Single Integrals

**4.1.** $k^2 = 1/2\left(1 - \frac{p^2-a^2+b^2}{\sqrt{(p^2-a^2+b^2)^2+4a^2b^2}}\right)$,

$k^2 + k'^2 = 1, \qquad p > a$

---

$$\int_0^\infty e^{-px}I_0(ax)J_0(bx)\,dx = \frac{2\sqrt{kk'}}{\pi\sqrt{ab}}K(k). \quad (1)$$

$$\int_0^\infty e^{-px}I_0(ax)J_0(bx)x\,dx = \frac{2p\sqrt{(kk')^3}}{\pi\sqrt{(ab)^3}}[2E(k) - K(k)]. \quad (2)$$

$$\int_0^\infty e^{-px}I_1(ax)J_1(bx)\,dx = \frac{2}{\pi\sqrt{abkk'}}[E(k) - (1-k^2)K(k)]. \quad (3)$$

$$\int_0^\infty e^{-px}I_1(ax)J_1(bx)x\,dx = \frac{2p\sqrt{kk'}}{\pi\sqrt{(ab)^3}}[(1-k^2)K(k) - (1-2k^2)E(k)]. \quad (4)$$

$$\int_0^\infty e^{-px}I_2(ax)J_2(bx)\,dx = \frac{2}{3\pi\sqrt{ab(kk')^3}}[(1-k^2)(2-3k^2)K(k) - 2(1-2k^2)E(k)]. \quad (5)$$

$$\int_0^\infty e^{-px}I_2(ax)J_2(bx)x\,dx = \frac{2p}{\pi\sqrt{(ab)^3kk'}}[2(1-k^2+k^4)E(k) - (1-k^2)(2-k^2)K(k)]. \quad (6)$$

$$\int_0^\infty e^{-px}I_3(ax)J_3(bx)\,dx = \frac{2}{15\pi\sqrt{ab(kk')^5}}[(8-23k^2+23k^4)E(k) - (1-k^2)(8-19k^2+15k^4)K(k)]. \quad (7)$$

$$\int_0^\infty e^{-px} I_3(ax) J_3(bx)\, x\, dx = \frac{2p}{3\pi \sqrt{(abkk')^3}} [(8-15k^2+3k^4)(1-k^2)K(k)$$

$$-(8-19k^2+9k^4-6k^6)E(k)]. \qquad (8)$$

## 4.2. $k^2 = \frac{a^2}{a^2+b^2}$

$$\int_0^\infty [I_0(ax) - \mathbf{L_0}(ax)]\, J_0(bx)\, dx = \frac{2k}{\pi a} K(k).$$ OB 26 (2.141) (1)

$$\int_0^\infty [I_1(ax) - \mathbf{L_1}(ax)]\, J_0(bx)\, x^{-1} dx = \frac{2}{\pi k} [K(k) - E(k)]. \qquad (2)$$

$$\int_0^\infty [I_2(ax) - \mathbf{L_2}(ax)]\, J_0(bx)\, x^{-2} dx = \frac{2a}{9\pi k^3} [(2+k^2)K(k) - 2(1+k^2)E(k)]. \qquad (3)$$

$$\int_0^\infty [I_3(ax) - \mathbf{L_3}(ax)]\, J_0(bx)\, x^{-3} dx = \frac{2a^2}{225\pi k^5} [(8+3k^2+4k^4)K(k) - (8+7k^2+8k^4)E(k)]. \qquad (4)$$

## 4.3. $k^2 = \left(\frac{a-b}{a+b}\right)^2$

$$\int_0^\infty [\mathbf{H_0}(ax) - Y_0(ax)]\, J_0(bx)\, dx = \frac{4}{\pi(a+b)} K(k).$$ OB 26 (2.140) (1)

## 4.4. $k^2 = \left(\frac{1-a}{1+a}\right)^2$

$$\int_0^\infty e^{-(1/2a^2-1)x^2} I_0\left(\frac{1}{2} a^2x^2\right) \operatorname{erfc}(x)\, x^2 dx = \frac{2}{(1+a)\sqrt{\pi}} K(k). \qquad (1)$$

$$\int_0^\infty e^{-(1/2a^2-1)x^2} \left[I_0\left(\frac{1}{2} a^2x^2\right) - I_1\left(\frac{1}{2} a^2x^2\right)\right] \operatorname{erfc}(x)\, x^2 dx = \frac{1}{a^2(1-a^2)\sqrt{\pi}} [(1+a)E(k) - 2aK(k)]. \qquad (2)$$

## 4.5. $k^2 = \frac{1}{1+a^2}$

$$\int_0^\infty e^{-1/2a^2x^2} K_0\left(\frac{1}{2} a^2x^2\right) \operatorname{erf}(x) x dx = \frac{1}{a^2\sqrt{1+a^2}} [(1+a^2)E(k) - a^2K(k)]. \qquad (1)$$

$$\int_0^\infty e^{-1/2a^2x^2} K_1\left(\frac{1}{2} a^2x^2\right) \operatorname{erf}(x) x^3 dx = \frac{1}{6a^4\sqrt{1+a^2}} [2a^2(a^2-2)K(k)$$

$$-(2a^4-3a^2-8)E(k)]. \qquad (2)$$

## 4.6. $k^2=\dfrac{1}{1+2a}$

$$\int_0^\infty e^{-(1+a)x^2}K_0(ax^2)\mathbf{L}_0(x^2)dx=\sqrt{\pi}kK(k). \quad (1)$$

$$\int_0^\infty e^{-(1+a)x^2}K_0(ax^2)\mathbf{L}_1(x^2)x^2dx=\frac{\sqrt{\pi}}{16a^2k}[2E(k)-(1-k^2)K(k)]. \quad (2)$$

$$\int_0^\infty e^{-(1+a)x^2}K_0(ax^2)\mathbf{L}_2(x^2)x^4dx=\frac{3\sqrt{\pi}}{512a^4k^3}[8(1+k^2)E(k)-(1-k^2)(5+3k^2)K(k)]. \quad (3)$$

## 4.7. $k^2=\dfrac{a}{1+a}$

$$\int_0^\infty e^{-(1+2a)x^2}K_0(x^2)L_1(2ax^2)\ x^2dx=\frac{\sqrt{\pi}\ k^3}{8\sqrt{2a^3}}[2E(k)-K(k)]. \quad (1)$$

$$\int_0^\infty e^{-(1+2a)x^2}K_0(x^2)L_2(2ax^2)\ x^4dx=\frac{3\sqrt{\pi}\ k^7}{128\sqrt{2a^7}}[8(1-a)E(k)-(5-3a)K(k)]. \quad (2)$$

$$\int_0^\infty e^{-(1+2a)x^2}K_0(x^2)L_3(2ax^2)\ x^6dx=\frac{5\sqrt{\pi}\ k^{11}}{1024\sqrt{2a^{11}}}$$
$$\times[2(23-82a+23a^2)E(k)-(31-82a+15a^2)K(k)]. \quad (3)$$

$$\int_0^\infty e^{-(1+2a)x^2}K_1(x^2)L_1^{(2)}(2ax^2)\ x^6dx=\frac{3\sqrt{\pi}\ k^7}{128\sqrt{2a^{11}}}$$
$$\times[(2+9a-a^2)K(k)-2(1-a)(1+6a+a^2)E(k)]. \quad (4)$$

$$\int_0^\infty e^{-(1+2a)x^2}K_1(x^2)L_2^{(2)}(2ax^2)\ x^8dx=\frac{15\sqrt{\pi}\ k^{11}}{2048\sqrt{2a^{15}}}$$
$$\times[(2+21a-108a^2+a^3)K(k)-2(1+11a-108a^2+11a^3+a^4)E(k)]. \quad (5)$$

## 4.8.[6] $k^2=\dfrac{b-a}{b+a}$, $b>a$

$$\int_0^\infty [J_{-1/2}(ax^2)Y_0(ax^2)+J_0(ax^2)Y_{-1/2}(ax^2)]J_0(2bx^2)x^2dx=-\sqrt{\frac{2}{\pi^3ab(a+b)}}K(k). \quad (1)$$

$$\int_0^\infty [J_{-1/2}(ax^2)Y_1(ax^2)+J_1(ax^2)Y_{-1/2}(ax^2)]J_1(2bx^2)x^2dx=-\frac{\sqrt{2(a+b)}}{\sqrt{\pi^3a^3b}}\left[E(k)-\frac{a}{a+b}K(k)\right]. \quad (2)$$

$$\int_0^\infty [J_{-1/2}(ax^2)Y_2(ax^2)+J_2(ax^2)Y_{-1/2}(ax^2)]J_2(2bx^2)x^2dx$$
$$=-\frac{1}{3}\sqrt{\frac{2}{\pi^3a^5b(a+b)}}[4b(a+b)E(k)-a(4b+a)K(k)]. \quad (3)$$

[6] For $b<a$ the integrals vanish.

### 4.9. $k^2 = \dfrac{b - \sqrt{b^2 - a^2}}{2b}, \qquad b > a$

$$\int_0^\infty J_{1/4}^2(ax^2) J_{1/4}^2(bx^2)\ x^2 dx = \frac{\Gamma(1/4)}{2\pi^2 b\ \sqrt{a}\ \Gamma(3/4)} K(k). \tag{1}$$

### 4.10. $k^2 = \dfrac{b - \sqrt{b^2 - a^2}}{2b}, \qquad b > a$

$$\int_0^\infty J_{-1/4}(ax^2) Y_{-1/4}(ax^2) J_{-1/4}(bx^2) Y_{-1/4}(bx^2)\ x^2 dx = \frac{\Gamma(1/4)}{2\pi^2 b\ \sqrt{a}\ \Gamma(3/4)} K(k). \tag{1}$$

$$\int_0^\infty J_{1/4}(ax^2) Y_{1/4}(ax^2) J_{1/4}(bx^2) Y_{1/4}(bx^2)\ x^4 dx = \frac{\Gamma(3/4)\ \sqrt{a}}{4\pi^2 b^3 \Gamma(1/4) k^2 (1-k^2)(1-2k^2)} \times [(1-k^2)K(k) - (1-2k^2)E(k)]. \tag{2}$$

## 5. Double Integrals

### 5.1. $k^2 = \dfrac{a^2}{pq}, \qquad pq > a^2$

$$\int_0^\infty \int_0^\infty e^{-px^2 - qy^2} I_0(2axy)\, dx dy = \frac{1}{2\sqrt{pq}} K(k). \tag{1}$$

$$\int_0^\infty \int_0^\infty e^{-px^2 - qy^2} I_0(2axy) y^2 dx dy = \frac{1}{4(1-k^2)\sqrt{pq^3}} E(k). \tag{2}$$

$$\int_0^\infty \int_0^\infty e^{-px^2 - qy^2} I_0(2axy) y^4 dx dy = \frac{1}{8(1-k^2)^2 \sqrt{pq^5}} [2(2-k^2)E(k) - (1-k^2)K(k)]. \tag{3}$$

$$\int_0^\infty \int_0^\infty e^{-px^2 - qy^2} I_0(2axy) y^6 dx dy = \frac{1}{16(1-k^2)^3 \sqrt{pq^7}} \times [(23 - 23k^2 + 8k^4)E(k) - 4(2-k^2)(1-k^2)K(k)]. \tag{4}$$

$$\int_0^\infty \int_0^\infty e^{-px^2 - qy^2} I_0(2axy) y^8 dx dy = \frac{1}{32(1-k^2)^4 \sqrt{pq^9}} \times [8(2-k^2)(11 - 11k^2 + 6k^4)E(k) - (1-k^2)(71 - 71k^2 + 24k^4)K(k)]. \tag{5}$$

$$\int_0^\infty \int_0^\infty e^{-px^2 - qy^2} I_0(2axy)\ x^2 y^2 dx dy = \frac{1}{8(1-k^2)^2\ \sqrt{p^3 q^3}} [2E(k) - (1-k^2)K(k)]. \tag{6}$$

$$\int_0^\infty \int_0^\infty e^{-px^2 - qy^2} I_0(2axy)\ x^2 y^4 dx dy = \frac{1}{16(1-k^2)^3\ \sqrt{p^3 q^5}} [(7+k^2)E(k) - 4(1-k^2)K(k)]. \tag{7}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)\ x^2y^6dxdy=\frac{1}{32(1-k^2)^4\sqrt{p^3q^7}}$$
$$[2(19+6k^2-k^4)E(k)-(1-k^2)(23+k^2)K(k)]. \tag{8}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)\ x^2y^8dxdy=\frac{1}{64(1-k^2)^5\sqrt{p^3q^9}}$$
$$\times[(281+142k^2-47k^4+8k^6)E(k)-4(1-k^2)(44+5k^2-k^4)K(k)]. \tag{9}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)'x^4y^4dxdy=\frac{3}{32(1-k^2)^4\sqrt{p^5q^5}}$$
$$\times[8(1+k^2)E(k)-(1-k^2)(5+3k^2)K(k)]. \tag{10}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)\ x^4y^6dxdy=\frac{3}{64(1-k^2)^5\sqrt{p^5q^7}}$$
$$\times[(43+82k^2+3k^4)E(k)-4(1-k^2)(7+9k^2)K(k)]. \tag{11}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)x^4y^8dxdy=\frac{3}{128(1-k^2)^6\sqrt{p^5q^9}}$$
$$\times[2(158+449k^2+36k^4-3k^6)E(k)-(1-k^2)(211+426k^2+3k^4)K(k)]. \tag{12}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)x^6y^6dxdy=\frac{15}{128(1-k^2)^6\sqrt{p^7q^7}}$$
$$\times[2(23+82k^2+23k^4)E(k)-(1-k^2)(31+82k^2+15k^4)K(k)]. \tag{13}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(2axy)x^6y^8dxdy=\frac{15}{256(1-k^2)^7\sqrt{p^7q^9}}$$
$$\times[(337+1773k^2+947k^4+15k^6)E(k)-8(1-k^2)(29+118k^2+45k^4)K(k)]. \tag{14}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(2axy)xydxdy=\frac{1}{4pqk(1-k^2)}[E(k)-(1-k^2)K(k)]. \tag{15}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(2axy)xy^3dxdy=\frac{1}{8pq^2k(1-k^2)^2}[(1+k^2)E(k)-(1-k^2)K(k)]. \tag{16}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(2axy)\ xy^5dxdy=\frac{1}{16pq^3k(1-k^2)^3}$$
$$\times[(3+7k^2-2k^4)E(k)-(1-k^2)(3+k^2)K(k)]. \tag{17}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(2axy)\ xy^7dxdy=\frac{1}{32pq^4k(1-k^2)^4}$$
$$\times[(15+58k^2-33k^4+8k^6)E(k)-(1-k^2)(15+13k^2-4k^4)K(k)]. \tag{18}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(2axy)\ x^3y^3dxdy=\frac{1}{16p^2q^2k(1-k^2)^3}$$
$$\times[(1+7k^2)E(k)-(1-k^2)(1+3k^2)K(k)]. \tag{19}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)\, x^3y^5dxdy = \frac{3}{32p^2q^3k(1-k^2)^4}$$
$$\times [(1+14k^2+k^4)E(k)-(1-k^2)(1+7k^2)K(k)]. \quad (20)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy)\, x^3y^7dxdy = \frac{3}{64p^2q^4k(1-k^2)^5}$$
$$\times [(5+108k^2+17k^4-2k^6)E(k)-(1-k^2)(5+58k^2+k^4)K(k)]. \quad (21)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy) x^5y^5dxdy = \frac{3}{64p^3q^3k(1-k^2)^5}$$
$$\times [(3+82k^2+43k^4)E(k)-(1-k^2)(3+46k^2+15k^4)K(k)]. \quad (22)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy) x^5y^7dxdy = \frac{15}{128p^3q^4k(1-k^2)^6}$$
$$\times [(3+125k^2+125k^4+3k^6)E(k)-(1-k^2)(3+74k^2+51k^4)K(k)]. \quad (23)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy) x^{-1}y^{-1}dxdy = \frac{1}{k}\,[E(k)-(1-k^2)K(k)]. \quad (24)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy) x^{-1}ydxdy = \frac{1}{2qk}\,[K(k)-E(k)]. \quad (25)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy) x^{-1}y^3dxdy = \frac{1}{4q^2k(1-k^2)}\,[(1-k^2)K(k)-(1-2k^2)E(k)]. \quad (26)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy) x^{-1}y^5dxdy = \frac{1}{8q^3k(1-k^2)^2}$$
$$\times [(1-k^2)(3-4k^2)K(k)-(3-13k^2+8k^4)E(k)]. \quad (27)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_1(2axy) x^{-1}y^7dxdy = \frac{1}{16q^4k(1-k^2)^3}\,[(1-k^2)(15-43k^2+24k^4)K(k)$$
$$-(15-103k^2+128k^4-48k^6)E(k)]. \quad (28)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_2(2axy) dxdy = \frac{1}{2k^2\sqrt{pq}}\,[(2-k^2)K(k)-2E(k)]. \quad (29)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_2(2axy) y^2dxdy = \frac{1}{4k^2(1-k^2)\sqrt{pq^3}}\,[(2-k^2)E(k)-2(1-k^2)K(k)]. \quad (30)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_2(2axy) y^4dxdy = \frac{1}{8k^2(1-k^2)^2\sqrt{pq^5}}$$
$$\times [2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \quad (31)$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} I_2(2axy) y^6dxdy = \frac{1}{16k^2(1-k^2)^3\sqrt{pq^7}} +$$

$$\times[(6-9k^2+19k^4-8k^6)E(k)-2(1-k^2)(3-3k^2+2k^4)K(k)]. \quad (32)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^2y^2dxdy=\frac{1}{8k^2(1-k^2)^2\sqrt{p^3q^3}}$$

$$\times[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]. \quad (33)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^2y^4dxdy=\frac{1}{16k^2(1-k^2)^3\sqrt{p^3q^5}}$$

$$\times[2(1-k^2)(1-3k^2)K(k)-(2-7k^2-3k^4)E(k)]. \quad (34)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^2y^6dxdy=\frac{3}{32k^2(1-k^2)^4\sqrt{p^3q^7}}$$

$$\times[(1-k^2)(2-9k^2-k^4)K(k)-2(1+k^2)(1-6k^2+k^4)E(k)]. \quad (35)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^4y^4dxdy=\frac{1}{32k^2(1-k^2)^4\sqrt{p^5q^5}}$$

$$\times[(1-k^2)(2-11k^2-15k^4)K(k)-2(1-6k^2-19k^4)E(k)]. \quad (36)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^4y^6dxdy=\frac{3}{64k^2(1-k^2)^5\sqrt{p^5q^7}}$$

$$\times[2(1-k^2)(1-8k^2-25k^4)K(k)-(2-17k^2-108k^4-5k^6)E(k)]. \quad (37)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^4y^8dxdy=\frac{15}{128k^2(1-k^2)^6\sqrt{p^5q^9}}$$

$$\times[(1-k^2)(2-21k^2-108k^4-k^6)K(k)-2(1-11k^2-108k^4-11k^6+k^8)E(k)]. \quad (38)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^{-2}dxdy=\frac{\sqrt{p}}{3k^2\sqrt{q}}[(2-k^2)E(k)-2(1-k^2)K(k)]. \quad (39)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^{-2}y^2dxdy=\frac{\sqrt{p}}{6k^2\sqrt{q^3}}[(2+k^2)K(k)-2(1+k^2)E(k)]. \quad (40)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^{-2}y^4dxdy=\frac{\sqrt{p}}{12k^2(1-k^2)\sqrt{q^5}}$$

$$\times[2(1-k^2)(1+2k^2)K(k)-(2+3k^2-8k^4)E(k)]. \quad (41)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^{-2}y^6dxdy=\frac{\sqrt{p}}{8k^2(1-k^2)^2\sqrt{q^7}}$$

$$\times[(1-k^2)(2+5k^2-8k^4)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \quad (42)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(2axy)x^{-2}y^{-2}dxdy=\frac{2\sqrt{pq}}{9k^2}[(2-3k^2)(1-k^2)K(k)-2(1-2k^2)E(k)]. \tag{43}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)xy\,dxdy=\frac{1}{4pqk^3(1-k^2)}$$
$$\times[(8-7k^2)E(k)-(1-k^2)(8-3k^2)K(k)]. \tag{44}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)xy^3dxdy=\frac{1}{8pq^2k^3(1-k^2)^2}$$
$$\times[(8-9k^2)(1-k^2)K(k)-(8-13k^2+3k^4)E(k)]. \tag{45}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)xy^5dxdy=\frac{1}{16pq^3k^3(1-k^2)^3}$$
$$\times[(1-k^2)(8-15k^2+3k^4)K(k)-(1-2k^2)(8-3k^2+3k^4)E(k)]. \tag{46}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^3y^3dxdy=\frac{1}{16p^2q^2k^3(1-k^2)^3}$$
$$\times[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]. \tag{47}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^3y^5dxdy=\frac{1}{32p^2q^3k^3(1-k^2)^4}$$
$$\times[(8-33k^2+58k^4+15k^6)E(k)-(1-k^2)(8-29k^2+45k^4)K(k)]. \tag{48}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^5y^5dxdy=\frac{1}{64p^3q^3k^3(1-k^2)^5}$$
$$\times[(8-47k^2+142k^4+281k^6)E(k)-(1-k^2)(8-43k^2+122k^4+105k^6)K(k)]. \tag{49}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-1}y\,dxdy=\frac{1}{6qk^3}[(8-5k^2)K(k)-(8-k^2)E(k)]. \tag{50}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-1}y^3dxdy=\frac{1}{12q^2k^3(1-k^2)}$$
$$\times[(8-3k^2-2k^4)E(k)-(1-k^2)(8+k^2)K(k)]. \tag{51}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-1}y^5dxdy=\frac{1}{24q^3k^3(1-k^2)^2}$$
$$\times[(1+k^2)(8-13k^2+8k^4)E(k)-(1-k^2)(8-k^2-4k^4)K(k)]. \tag{52}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-1}y^{-1}dxdy=\frac{1}{9k^3}[(8-7k^2)E(k)-(8-3k^2)(1-k^2)K(k)]. \tag{53}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-1}y^{-3}dxdy=\frac{2q}{45k^3}[(8-9k^2)(1-k^2)K(k) - (8-13k^2+3k^4)E(k)]. \quad (54)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-3}y\,dxdy=\frac{p}{15qk^3} \times[(8-3k^2-2k^4)E(k)-(8+k^2)(1-k^2)K(k)]. \quad (55)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-3}y^3dxdy=\frac{p}{30q^2k^3}[(8+3k^2+4k^4)\ K(k)-(8+7k^2+8k^4)E(k)]. \quad (56)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-3}y^5dxdy=\frac{p}{60q^3k^3(1-k^2)} \times[(1-k^2)(8+13k^2+24k^4)K(k)-(8+9k^2+16k^4-48k^6)E(k)]. \quad (57)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(2axy)x^{-3}y^{-3}dxdy=\frac{4pq}{225k^3}[(8-23k^2+23k^4)\ E(k) - (1-k^2)(8-19k^2+15k^4)K(k)]. \quad (58)$$

## 5.2. $k^2=\dfrac{\sqrt{pq}-\sqrt{pq-a^2}}{2\sqrt{pq}}$, $pq>a^2$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0^2(axy)dxdy=\frac{1}{\pi\sqrt{pq}}[K(k)]^2. \quad (1)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0^2(axy)x^2dxdy=\frac{1}{2\pi(1-2k^2)\sqrt{p^3q}}K(k)\ [2E(k)-K(k)]. \quad (2)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1^2(axy)dxdy=\frac{1}{\pi k^2(1-k^2)\sqrt{pq}}[E(k)-\ 1-k^2)K(k)]. \quad (3)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1^2(axy)x^2dxdy=\frac{1}{2\pi k^2(1-k^2)(1-2k^2)\sqrt{p^3q}} \times[E(k)-(1-k^2)K(k)][(1-k^2)K(k)-(1-2k^2)E(k)]. \quad (4)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1^2(axy)x^{-2}y^2dxdy=\frac{\sqrt{p}}{3\pi k^2(1-k^2)\sqrt{q^3}}\times[(1-k^2)K(k^1)-(1-2k^2)E(k)]^2 \quad (5)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1^2(axy)x^{-2}y^4dxdy=\frac{\sqrt{p}}{6\pi k^2(1-k^2)(1-2k^2)\sqrt{q^5}}[(1-16k^2+16k^4)E(k) - (1-k^2)(1-8k^2)K(k)]\ [(1-2k^2)E(k)-(1-k^2)K(k)]. \quad (6)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2^2(axy)\,dxdy=\frac{1}{9\pi k^4(1-k^2)^2\sqrt{pq}}[(2-3k^2)\times(1-k^2)K(k)-2(1-2k^2)E(k)]^2. \tag{7}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2^2(axy)x^2\,dxdy=\frac{1}{6\pi k^4(1-k^2)^2(1-2k^2)\sqrt{p^3q}}[2(1-k^2+k^4)E(k)-(2-k^2)(1-k^2)K(k)]\,[(2-3k^2)(1-k^2)K(k)-2(1-2k^2)E(k)]. \tag{8}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2^2(axy)x^{-2}y^2\,dxdy=\frac{\sqrt{p}}{15\pi k^4(1-k^2)^2\sqrt{q^3}}\times[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]^2. \tag{9}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3^2(axy)\,dxdy=\frac{1}{225\pi k^6(1-k^2)^3\sqrt{pq}}\times[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]^2. \tag{10}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3^2(axy)x^{-2}y^2\,dxdy=\frac{\sqrt{p}}{315\pi k^6(1-k^2)^3\sqrt{q^3}}\times[(1-2k^2)(8-3k^2+3k^4)E(k)-(1-k^2)(8-15k^2+3k^4)K(k)]^2. \tag{11}$$

## 5.3. $k^2=\dfrac{\sqrt{pq}-\sqrt{pq-a^2}}{2\sqrt{pq}},\qquad pq>a^2$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(axy)I_0(axy)xy\,dxdy=\frac{1-2k^2}{2\pi pqk\sqrt{1-k^2}}K(k)\,[E(k)-(1-k^2)K(k)]. \tag{1}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(axy)I_0(axy)x^{-1}y^3\,dxdy=\frac{1-2k^2}{2\pi q^2k\sqrt{1-k^2}}\times[2E(k)-K(k)][(1-k^2)K(k)-(1-2k^2)E(k)]. \tag{2}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(axy)I_1(axy)xy\,dxdy=\frac{1-2k^2}{2\pi pqk^3\sqrt{(1-k^2)^3}}\times[E(k)-(1-k^2)K(k)]\,[(2-3k^2)(1-k^2)K(k)-2(1-2k^2)E(k)]. \tag{3}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(axy)I_1(axy)x^{-1}y^3\,dxdy=\frac{1-2k^2}{6\pi q^2k^3\sqrt{(1-k^2)^3}}\times[(1-k^2)K(k)-(1-2k^2)E(k)]\,[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \tag{4}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(axy)I_2(axy)xydxdy$$

$$=\frac{1-2k^2}{18\pi pqk^5\sqrt{(1-k^2)^5}}[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]+$$

$$\times[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]. \quad (5)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(axy)I_2(axy)x^{-1}y^3dxdy=\frac{1-2k^2}{30\pi q^2k^5\sqrt{(1-k^2)^5}}[2(1-k^2+k^4)E(k)$$

$$-(1-k^2)(2-k^2)K(k)][(1-k^2)(8-15k^2+3k^4)K(k)-(1-2k^2)(8-3k^2+3k^4)E(k)]. \quad (6)$$

## 5.4. $k^2=\dfrac{4abc^2}{[(p+a)(q-b)-c^2][(p-a)(q+b)-c^2]}$

$(p-a)(q-b)>c^2, \quad p>a, \quad q>b$

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$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(ax^2)I_0(by^2)I_0(2cxy)\ xydxdy=\frac{k}{4\pi c\sqrt{ab}}K(k). \quad (1)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(ax^2)I_0(by^2)I_0(2cxy)\ xy^3dxdy=\frac{k^3(qp^2-qa^2-pc^2)}{16\pi(1-k^2)c^3\sqrt{(ab)^3}}E(k). \quad (2)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_0(ax^2)I_1(by^2)I_0(2cxy)\ xy^3dxdy$$

$$=\frac{k^3[(qp^2-qa^2-pc^2)^2+b^2(p^2-a^2)^2-a^2c^4]}{32\pi(1-k^2)(p^2-a^2)c^3\sqrt{a^3b^5}}E(k)-\frac{k}{8\pi c\sqrt{ab^3}}K(k). \quad (3)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(ax^2)I_1(by^2)I_2(2cxy)\ xydxdy=\frac{1}{4\pi kc\sqrt{ab}}[(2-k^2)K(k)-2E(k)]. \quad (4)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_1(ax^2)I_1(by^2)I_2(2cxy)xy^3dxdy=\frac{k(qp^2-qa^2-pc^2)}{16\pi(1-k^2)c^3\sqrt{(ab)^3}}$$

$$\times[(2-k^2)E(k)-2(1-k^2)K(k)]. \quad (5)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_2(ax^2)I_2(by^2)I_4(2cxy)xydxdy=\frac{1}{12\pi k^3c\sqrt{ab}}$$

$$\times[(4-k^2)(4-3k^2)K(k)-8(2-k^2)E(k)]. \quad (6)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_3(ax^2)I_3(by^2)I_6(2cxy)xydxdy=\frac{1}{60\pi k^5c\sqrt{ab}}$$

$$\times[(128-128k^2+15k^4)(2-k^2)K(k)-2(128-128k^2+23k^4)E(k)]. \quad (7)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}I_4(ax^2)I_4(by^2)I_8(2cxy)xydxdy=\frac{1}{420\pi k^7c\sqrt{ab}}$$

$$\times[(6144-12288k^2+8000k^4-1856k^6+105k^8)K(k)-32(2-k^2)(96-96k^2+11k^4)E(k)]. \quad (8)$$

## 5.5. $k^2 = \dfrac{\sqrt{pq} - \sqrt{pq - b^2}}{2\sqrt{pq}}, \qquad pq > b^2$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_0(2\sqrt{pq}xy) I_0(4bxy)\, dxdy = \frac{k\sqrt{1-k^2}}{b} [K(k)]^2. \tag{1}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_0(2\sqrt{pq}xy) I_0(4bxy) x^2 dxdy = \frac{k\sqrt{1-k^2}}{4pb(1-2k^2)} K(k)[2E(k) - K(k)]. \tag{2}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_0(2\sqrt{pq}xy) I_1(4bxy) xy\, dxdy = \frac{1}{4b\sqrt{pq}(1-2k^2)} K(k)[E(k) - (1-k^2)K(k)]. \tag{3}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_0(2\sqrt{pq}xy) I_2(4bxy)\, dxdy = \frac{1}{bk\sqrt{1-k^2}} [E(k) - (1-k^2)K(k)]^2. \tag{4}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-by^2} K_0(2\sqrt{pq}xy) I_2(4bxy) x^2 dxdy = \frac{1}{4pbk(1-2k^2)\sqrt{1-k^2}}$$
$$\times [E(k) - (1-k^2)K(k)]\,[(1-k^2)K(k) - (1-2k^2)E(k)]. \tag{5}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_0(2\sqrt{pq}\, xy) I_3(4bxy)\, xy\, dxdy = \frac{1}{4b\sqrt{pq}\, k^2(1-k^2)(1-2k^2)}$$
$$\times [E(k) - (1-k^2)K(k)]\,[(1-k^2)(2-3k^2)K(k) - 2(1-2k^2)E(k)]. \tag{6}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_0(2\sqrt{pq}\, xy) I_4(4bxy)\, dxdy = \frac{1}{9bk^3\sqrt{(1-k^2)^3}}$$
$$\times [(1-k^2)(2-3k^2)K(k) - 2(1-2k^2)E(k)]^2. \tag{7}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_0(2\sqrt{pq}\, xy) I_4(4bxy)\, x^2 dxdy = \frac{1}{12pbk^3(1-2k^2)\sqrt{(1-k^2)^3}}$$
$$\times [(2-3k^2)(1-k^2)K(k) - 2(1-2k^2)E(k)]\,[2(1-k^2+k^4)E(k) - (1-k^2)(2-k^2)K(k)]. \tag{8}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_1(2\sqrt{pq}\, xy) I_0(4bxy)\, xy\, dxdy = \frac{k\sqrt{1-k^2}}{4b\sqrt{pq}\,(1-2k^2)} K(k)[2E(k) - K(k)]. \tag{9}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_1(2\sqrt{pq}\, xy) I_1(4bxy)\, x^2 dxdy = \frac{1}{4pb(1-2k^2)} [2E(k) - K(k)]$$
$$\times [(1-k^2)K(k) - (1-2k^2)E(k)]. \tag{10}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_1(2\sqrt{pq}\, xy) I_2(4bxy)\, xy\, dxdy = \frac{1}{4bk(1-2k^2)\sqrt{pq(1-k^2)}}$$
$$\times [E(k) - (1-k^2)K(k)\} \times [(1-k^2)K(k) - (1-2k^2)E(k)]. \tag{11}$$

$$\int_0^\infty \int_0^\infty e^{-px^2-qy^2} K_1(2\sqrt{pq}\, xy) I_2(4bxy)\, x^{-1} y\, dxdy = \frac{\sqrt{p}}{3bk\sqrt{q(1-k^2)}}$$
$$+ [(1-2k^2)E(k) - (1-k^2)K(k)]^2. \tag{12}$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_1(2\sqrt{pq}\,xy)I_2(4bxy)\;x^{-1}y^3dxdy=\frac{\sqrt{p}}{12bk(1-2k^2)\sqrt{q^3(1-k^2)}}$$

$$\times[(1-k^2)K(k)-(1-2k^2)E(k)]\,[(1-k^2)(1-8k^2)K(k)-(1-16k^2+16k^4)E(k)]. \quad (13)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_1(2\sqrt{pq}\,xy)I_4(4bxy)\;xydxdy=\frac{1}{12bk^3(1-2k^2)\sqrt{pq(1-k^2)^3}}\,[(2-3k^2)$$

$$\times(1-k^2)K(k)-2(1-2k^2)E(k)]\,[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \quad (14)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_2(2\sqrt{pq}\,xy)I_2(4bxy)\;x^2dxdy=\frac{1}{12pbk(1-2k^2)\sqrt{1-k^2}}\,[(1-k^2)K(k)$$

$$-(1-2k^2)E(k)]\,[(1-8k^2)(1-k^2)K(k)-(1-16k^2+16k^4)E(k)]. \quad (15)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_2(2\sqrt{pq}\,xy)I_4(4bxy)\;x^{-2}y^4dxdy=\frac{p}{420bq^2\,k^3(1-2k^2)\sqrt{(1-k^2)^3}}\,[(2+15k^2$$

$$-144k^4+128k^6)(1-k^2)K(k)-2(1+7k^2-135k^4+256k^6-128k^8)E(k)]$$

$$\times[(1-k^2)(2+5k^2-8k^4)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \quad (16)$$

$$\int_0^\infty\int_0^\infty e^{-px^2-qy^2}K_3(2\sqrt{pq}\,xy)I_4(4bxy)\;x^{-1}y^3dxdy=\frac{\sqrt{p}}{420bk^3(1-2k^2)\sqrt{q^3(1-k^2)^3}}$$

$$\times[(2+15k^2-144k^4+128k^6)\,(1-k^2)K(k)-2(1+7k^2-135k^4+256k^6-128k^8)E(k)]$$

$$\times\,[(1-k^2)(2+5k^2-8k^4)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \quad (17)$$

### 5.6. $k^2=\dfrac{4pqa}{(pq+a)^2-b^2}$, $\quad pq>a+b$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_0(2axy)\cosh(2bxy)\;dxdy=\frac{k}{4\sqrt{pqa}}K(k). \quad (1)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_1(2axy)\cosh(2bxy)\;x^{-1}ydxdy=\frac{\sqrt{p}}{4k\sqrt{q^3a}}[(2-k^2)K(k)-2E(k)]. \quad (2)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_2(2axy)\cosh(2bxy)\;x^{-2}y^2dxdy=\frac{\sqrt{p^3}}{12k^3\sqrt{q^5a}}$$

$$\times[(4-k^2)(4-3k^2)K(k)-8(2-k^2)E(k)]. \quad (3)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_3(2axy)\cosh(2bxy)\;x^{-3}y^3dxdy=\frac{\sqrt{p^5}}{60k^5\sqrt{q^7a}}$$

$$[(2-k^2)(128-128k^2+15k^4)K(k)-2(128-128k^2+23k^4)E(k)]. \quad (4)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_4(2axy)\cosh(2bxy)\ x^{-4}y^4dxdy=\frac{\sqrt{p^7}}{420k^7\sqrt{q^9a}}$$

$$\times[(6144-12288k^2+8000k^4-1856k^6+105k^8)K(k)-32(2-k^2)(96-96k^2+11k^4)E(k)]. \qquad (5)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_0(2axy)\sinh(2bxy)\ xydxdy=\frac{bk^3}{32(1-k^2)\sqrt{(pqa)^3}}E(k). \qquad (6)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_1(2axy)\sinh(2bxy)\ x^2dxdy=\frac{bk}{32(1-k^2)\sqrt{p^5qa^3}}$$

$$\times[(2-k^2)E(k)-2(1-k^2)K(k)]. \qquad (7)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_2(2axy)\sinh(2bxy)\ x^{-1}y^3dxdy=\frac{b\sqrt{p}}{32k(1-k^2)\sqrt{q^7a^3}}$$

$$\times\ [(16-16k^2+k^4)E(k)-8(1-k^2)(2-k^2)K(k)]. \qquad (8)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}I_3(2axy)\sinh(2bxy)\ x^{-2}y^4dxdy=\frac{b\sqrt{p^3}}{96k^3(1-k^2)\sqrt{q^9a^3}}$$

$$\times\ [(2-k^2)(128-128k^2+3k^4)E(k)-2(1-k^2)(128-128k^2+27k^4)K(k)]. \qquad (9)$$

**5.7.** $k_1^2=\dfrac{(pq-a)^2-b^2}{(pq+a)^2-b^2}, \quad k_2^2=\dfrac{b^2-(pq-a)^2}{4pqa}$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}K_0(2axy)\cosh(2bxy)\ dxdy=\frac{\pi\sqrt{1-k_1^2}}{4\sqrt{pqa}}K(k_1), \qquad b<|pq-a|;$$

$$=\frac{\pi}{4\sqrt{pqa}}K(k_2), \qquad |pq-a|<b<pq+a. \qquad (1)$$

$$\int_0^\infty\int_0^\infty e^{-p^2x^2-q^2y^2}K_0(2axy)\sinh(2bxy)\ xydxdy$$

$$=\frac{\pi b\sqrt{(1-k_1^2)^3}}{32k_1^2\sqrt{(pqa)^3}}[K(k_1)-E(k_1)], \qquad b<|pq-a|;$$

$$=\frac{\pi b}{32k_2^2(1-k_2^2)\sqrt{(pqa)^3}}[E(k_2)-(1-k_2^2)K(k_2)], \qquad |pq-a|<b<pq+a. \qquad (2)$$

$$\int_0^\infty \int_0^\infty e^{-p^2x^2-q^2y^2} K_1(2axy) \sinh(2bxy)\ x^2 dxdy$$

$$= \frac{\pi b\sqrt{1-k_1^2}}{32k_1^2\sqrt{p^5qa^3}} [(1+k_1^2)E(k_1) - (1-k_1^2)K(k_1)], \quad b < |pq-a|;$$

$$= \frac{\pi b}{32k_2^2(1-k_2^2)\sqrt{p^5qa^3}} [(1-k_2^2)K(k_2) - (1-2k_2^2)E(k_2)],$$

$$|pq-a| < b < pq+a. \qquad (3)$$

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## 6. References

For preparation of this paper the author referred to general expressions of these types of integrals in terms of hypergeometric functions or Legendre functions. The section numbers and the general expressions referred there are:

2.1 OF 63,167, OB 50 (6.10),(6.11), ET I 45 (13), 100 (11); 2.2 OF 65,170; 2.4 OF 65; 2.5 OF 169; 2.7 OF 69; 2.8 OF 69; 2.10 OF 174, ET II 348 (65); 2.12 OF 87,184, ET I 49 (42),106 (50); 2.13 OF 89,185, ET I 107 (59),(60); 2.14 OF 88,185; 2.15 OF 89,186; 2.16 OF 88,185; ET I 50 (51),107 (61); 2.18 OF 88,185, ET I 49 (47),106 (54); 2.19 LU 315 (13); 2.20 OF 88,185; 4.2 OB 125 (14.11); 4.4 EG 18 (19); 4.5 ET II 410 (43); 4.6 ET II 387 (13); 4.7 ET II 370 (44); 4.8 ET II 325 (12); 4.9 ET II 351 (10); 4.10 ET II 352 (15); 5.1 NM II 33 (137); 5.6 OS 624 (7).

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LU Luke, Y. L., Integrals of Bessel Functions (McGraw-Hill Book Co., Inc., New York 1962).

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## 8. Supplementary References

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