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Properties of Neighboring Sequences in Stratifiable Spaces*

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In T_0-spaces metrizability can be characterized in terms of mutual convergence of "neighboring sequences." In this paper Nagata spaces are characterized in terms of a convergence property of neighboring sequences and more generally it is shown that in all stratifiable spaces, neighboring sequences satisfy a similar convergence property.

Key words: Coconvergent; contraconvergent; Nagata spaces; open neighborhood assignments; stratifiable spaces; U-linked sequences

An open neighborhood assignment (ONA) is defined in [4][1] as a function

$$U : X \times Z \rightarrow \{N(x) : x \epsilon X\}$$

such that $x \epsilon U(x, n) \equiv U_n(x)$ where X is a topological space, Z is the set of natural numbers and $N(x)$ is the collection of open neighborhoods of x. If U is an ONA then the sequence $\{x_n\}$ is U-*linked* to $\{y_n\}$ if $x_n \epsilon U_n(y_n)$ for all n. Using the notation $Cp\{x_n\}$ for the set of cluster points of $\{x_n\}$ a space will be called *coconvergent* (*contraconvergent*) if $Cp\{x_n\} \subset Cp\{y_n\}$ ($Cp\{y_n\} \subset Cp\{x_n\}$) whenever $\{y_n\}$ is U-linked to $\{x_n\}$. If on X there is an ONA U satisfying some property P I shall say "X is P" or "U is P." Also, U will be said to be nested if $U_{n+1}(x) \subset U_n(x)$ for all x and n.

It was proved in [4]:

THEOREM 1: X *is metrizable iff it is a coconvergent, contraconvergent* T_0-*space.*

Also, examples of a T_2, *coconvergent space and a* T_2, *contraconvergent space, neither of which are metrizable were given in* [4]. *Coconvergence implies first countability whereas nonfirst countable contraconvergent spaces exist.*

R. W. Heath in [3] proved:

THEOREM 2: *A* T_1-*space* X *is a Nagata space* (*first countable and stratifiable*) *iff there is an ONA* U *on* X *such that* (*a*) U *is first countable and* (*b*) *for every* $x \epsilon X$ *and open set* R *containing* $x \epsilon X$ *there is an* $n \epsilon Z$ *such that* $U_n(x) \cap U_n(y)$ *implies* $y \epsilon R$.

(*Note: Condition* (*a*) *is implied by* (*b*)*: If* $x \epsilon X$ *and* $R \epsilon N(x)$ *such that for all* n *there is a* $y_n \epsilon U_n(x) - R$, *then* $U_n(x) \cap U_n(y_n) \neq 0$. *It follows from* (*b*) *there is a* $y_k \epsilon R$ *for some* k *contradicting the way the* y_n *were chosen.*)

PROPOSITION 3: *A* T_1-*space* X *is a Nagata space iff it is first countable and contraconvergent.*

PROOF: Let U be an ONA on X satisfying the conditions of Theorem 2. Without loss of generality we may take U to be nested. Let $\{y_n\}$ be U-linked to $\{x_n\}$ with $y \epsilon Cp\{y_n\}$. Given an $R \epsilon N(y)$ and $N \epsilon Z$ there is an $n_1 > N$ such that $U_{n_1}(y) \subset R$. Also, since U is nested there is an $n_2 > n_1$ such that if x satisfies $U_{n_2}(x) \cap U_{n_2}(y) \neq 0$ then $x \epsilon R$. Finally, there is a $k > n_2$ such

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[1] Figures in brackets indicate the literature references at the end of this paper.

that $y_k \epsilon U_{n_2}(y) \subset U_{n_1}(y)$. Since $\{y_n\}$ is U-linked to $\{x_n\}$, $y_k \epsilon U_k(x_k) \cap U_{n_2}(y)5 \subset U_{n_2}(x_k) \cap U_{n_2}(y)$. It now follows from $k > N$ $y \epsilon Cp\{x_n\}$ and that U is contraconvergent.

Conversely, without loss of generality it may be assumed there is an ONA U on X which is contraconvergent and first countable. If U does not satisfy the condition of Theorem 2, then for some x and $R \epsilon N(x)$ there are sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \epsilon U_n(x) \cap U_n(y_n)$ and $y_n \notin R$ for all n. It follows from the first countability and contraconvergence of U that $x \epsilon Cp\{y_n\}$ which is a contradiction.

COROLLARY 4: *A* T_0*-space is metrizable iff it is coconvergent and stratifiable.*

Coconvergent spaces are those included in the class of spaces in which compact sets have countable local bases (D_0*-spaces*) [5]. *The set of all ordinals less than or equal to the first uncountable ordinal, with the order topology is an example of a* D_0*-space which is not coconvergent. It was shown in* [5] *that spaces in which the stratifications of open sets satisfy a certain monotone condition, "coconvergent" can be replaced by "*D_0*" in Corollary 4.*

If X *is stratifiable, I will use:* R_n, $(X-F)_n$ *and* $[U_k(x)]_n$ *to denote the* nth *layers of stratifications of the open sets* R, X−F *and* $U_k(x)$ *respectively and more generally,* A^- *for the closure of* A. *Without loss of generality I shall assume* $R_n \subset R_{n+1}$ *for any open* R. *It will be shown that if* X *is stratifiable that the ONA* U *defined by* $U_n(x) = X - (X - x_n)^-$ *satisfies a condition similar to that defining contraconvergent spaces. The ONA* U *will be referred to as the ONA associated with the given stratification.*

DEFINITION 5: *An ONA* U *on* X *satisfies property* A *if whenever* $\{y_n\}$ *is* U*-linked to* $\{x_n\}$ *and* $y \epsilon Cp\{y_n\}$ *then for any* $N \epsilon Z$ *there is a* $k > N$ *such that* $x_k \epsilon U_N(y)$.

PROPOSITION 6: *A* T_1*-space is a Nagata space iff there is a first countable ONA,* U *on* X *satisfying property* A.

PROOF: If X is a Nagata space, by Proposition 3 there is an ONA, U which is first countable and contraconvergent. If $\{y_n\}$ is U-linked to $\{x_n\}$ and $y \epsilon Cp\{y_n\}$ then $y \epsilon Cp\{x_n\}$ and it follows that U satisfies condition A.

Let U be a first countable ONA satisfying condition A. If $\{y_n\}$ is U-linked to $\{x_n\}$ with $y \epsilon Cp\{y_n\}$ and if $N_1 \epsilon Z$ and $R \epsilon N(y)$ then there is an $N_2 > N_1$ and by property A a $k > N_2$ such that $x_k \epsilon U_{N_2}(y) \subset R$ proving $y \epsilon Cp\{x_n\}$ and that U is contraconvergent. By Proposition 3, X is a Nagata space.

PROPOSITION 7: *Let* X *be a stratifiable space. Then the ONA* U *associated with a given stratification satisfies property* A.

PROOF: It follows from $R_n \subset R_{n+1}$ for any open R that U is nested. If U does not satisfy property A, then there exist a $\{y_n\}$ U-linked to $\{x_n\}$, a $y \epsilon Cp\{y_n\}$ and an $N \epsilon Z$ such that for all $k > N$, $x_k \notin U_N(y)$. Hence $F \equiv \{x_k : k > N\}^- \subset X - U_N(y)$. For all $n \epsilon Z$, $[U_N(y)]_n^- \subset (X-F)_n^-$. Also, for all $k > N$, $x_k \epsilon F$ and for all $n \epsilon Z$, $(X-F)_n^- \subset (X - x_k)_n^-$. Hence for all $k > N$ and $n \epsilon Z$, $U_n(x_k) \subset X - (X-F)_n^-$. Since $y \epsilon U_N(y)$, it follows $y \epsilon [U_N(y)]_M$ for some M. Therefore there is an $r \epsilon Z$ with $r > \max(N, M)$ such that $y_r \epsilon [U_N(y)]_M$. On the other hand we have $y_r \epsilon U_r(x_r) \subset U_M(x_r) \subset X - (X-F)_M^- \subset X - [U_N(y)]_M^-$ which is a contradiction.

In the next proposition I use Ceder's result [2] that locally compact M_3 ($\equiv$ stratifiable) spaces are metrizable.

PROPOSITION 8: *A locally compact space is metrizable iff it is contraconvergent and Hausdorff.*

PROOF: Let U be a contraconvergent ONA on X and for each x let $C(x)$ be a neighborhood of x whose closure is compact. Also for each x and n let $U_n'(x) = U_n(x) \cap C(x)$. Suppose $R \epsilon N(x)$ and for all n there is a $y_n \epsilon U_n'(x) - R$. Then there is a $y \epsilon [C(x)]^- \cap Cp\{y_n\}$. It follows that $y = x$ which is a contradiction. Hence U' is first countable and by Proposition 3 and Ceder's result, X is metrizable.

The converse is immediate from Theorem 1.

References

[1] Borges, C. J. R., On stratifiable spaces, Pacific J. Math. **3,** 175–186 (1966).

[2] Ceder, J. G., Some generalizations of metric spaces, Pacific J. Math. **11,** 105–125 (1961).

[3] Heath, R. W., On open mappings and certain spaces satisfying the first countability axioms, Fund. Math. **57,** 91–96 (1965).
[4] Sabella, R. R., Convergence properties of neighboring sequences, Proc. Amer. Math. Soc. **38,** 405–409 (1973).
[5] Sabella, R. R., Spaces in which compact sets have countable local bases, Proc. Amer. Math. Soc. (to appear).

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