JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 79B, Nos. 1 and 2, January-June 1975

# A Note on the Metrizability of Spaces With Countably Based Closed Sets

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**(January 14, 1975)**

The main result of this note is a generalization of an earlier theorem on the metrizability of spaces with countably based closed sets. Use is made of some results related to co-convergent spaces which are spaces having countably based compact sets.

Key words: Co-convergent; contra-convergent; Nagata spaces; open neighborhood assignments; stratifiable spaces; U-linked sequences.

C. E. Aull in [1][1] defines a $D_1$-*space* as a topological space in which every closed set $F$ has a countable local base $\{B_n(F)\}$: i.e., for each $n \in N$, the set of positive integers, $B_n(F)$ is an open set containing $F$; and if $F \subset R$, where $R$ is open, then $B_k(F) \subset R$ for some $k \in N$.

An *open neighborhood assignment* (ONA) is defined in [2] as a function

$$U : X \times N \to \cup \{N(x) : x \in X\}$$

such that $x \in U(x,n) \equiv U_n(x)$ where $X$ is a topological space and $N(x)$ is the collection of open neighborhoods of $x$. If $U$ is an ONA then the sequence $\{y_n\}$ is *U-linked* to $\{x_n\}$ if $y_n \in U_n(x_n)$ for all $n$. Using the notation $Cp\{x_n\}$ for the set of cluster points of $\{x_n\}$ we define a space to be *co-convergent* (*contra-convergent*) if $Cp\{x_n\} \subset Cp\{y_n\}$ ($Cp\{y_n\} \subset Cp\{x_n\}$) whenever $\{y_n\}$ is $U$-linked to $\{x_n\}$. If on $X$ there is an ONA $U$ having some property $P$ we shall say "$X$ is $P$" or "$U$ is $P$." Finally for any $S \subset X$ and ONA $U$ we have $U_n(S) \equiv \cup \{U_n(x) : x \in S\}$.

The following two theorems were proved in [2]:

THEOREM 1: X *is metrizable iff it is a co-convergent, contra-convergent* $T_0$-*space.*

THEOREM 2: *The following are equivalent on a space* X:

*(a)* X *is co-convergent.*

*(b) There exists an ONA* U *on* X *such that for any countably compact* K *and open* R *containing* K, $U_n(K) \subset R$ *for some* $n \in N$.

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*AMS (MOS) Subject Classification (1970):* Primary 54E35; Secondary 54D99.

[1] Figures in brackets indicate the literature references at the end of this paper.

(c) *There exists an ONA* U *on* X *such that for any convergent sequence* $\{x_n\}$ *with limit* $x_0$ *and open* R *containing* $\{x_k : k = 0,1,\ldots\}$, $U_n(\{x_k : k = 0,1,\ldots\}) \subset R$ *for some* $n \in N$.

A characterization of a *natural*-$D_1$ space is as a space $X$ on which there is an ONA $U$ such that for every closed set $F$, $\{U_n(F)\}$ is a local base for $F$. We note that if $X$ is $T_2$ and $U$ is a natural-$D_1$ ONA then it is co-convergent.

We have an analogue of Theorem 2:

THEOREM 3: *Let* U *be an ONA on* X. *Then the following are equivalent:*

(a) U *is natural*-$D_1$.

(b) *For any sequence* $\{x_k\}$, $\{U_n(\Delta\{x_k\})$ *is a local base for* $\Delta\{x_k\}$ *where* $\Delta\{x_k\} \equiv Cl\{x_n : n = 1,2,\ldots\}$.

PROOF: Only (b) implies (a) requires any consideration. Suppose $F$ is closed and is contained in an open $R$. If for each $n$ there is a $z_n \in U_n(F) - R$ then there is a sequence $\{x_n\}$ in $F$ with $\{z_n\}$ $U$-linked to $\{x_n\}$. But then $\Delta\{x_n\} \subset F$ and for some $k$, $U_k(\Delta\{x_n\}) \subset R$ implying the contradiction $z_k \notin R$.

The following result was proved in [3]:

THEOREM 4: *If* X *has at most a finite number of isolated points it is compact and metrizable iff it is natural*-$D_1$ *and Hausdorff.*

We shall generalize Theorem 4, by using Aull's result in [1] that every regular, $D_1$-space is the union of a countably compact set and a set of isolated points.

THEOREM 5: X *is compact and metrizable iff it is natural*-$D_1$ *and Hausdorff,*

PROOF: We need only consider the sufficiency part of the proof, the necessary part being the same as for Theorem 4.

Let $U$ be a natural-$D_1$ ONA on $X$. Without loss of generality we can assume $U$ is nested. Furthermore $\{U_n(x)\}$ is a local base for each $x \in X$. $X$ is regular, for if $F$ is closed and $x \in F$ and if for all $n$ there is a $z_n \in U_n(x) - U_n(F)$, then there is a sequence $\{y_n\}$ in $F$ such that $\{z_n\}$ is $U$-linked to $\{y_n\}$. It follows that $\{z_n\}$ converges to $x$. Hence there is an $M > 0$ such that for all $k > M$, $z_k \in X - F$. Since $U$ is natural-$D_1$ there is an $n > M$ such that $U_n(F) \subset X - \Delta\{z_k\}_{k=M}^{\infty}$, which implies $U_n(y_n) \subset X - \Delta\{z_k\}_{k=M}^{\infty}$, contradicting $z_n \in U_n(y_n)$.

Let $X = C \cup I$ where $C$ is countably compact and $I$ is a set of isolated points of $X$. We can assume $C \cap I = 0$. Again without loss of generality we can let $U_n(x) = \{x\}$ for all $x \in I$ and all $n$. Let $\{y_n\}$ be $U$-linked to $\{x_n\}$ and $y \in Cp\{y_n\}$. Then there is a subsequence $\{y_{n_k}\}$ **of** $\{y_n\}$ converging to $y$. If $x_{n_k} \in I$ for infinitely many $k$, then $\{x_{n_k}\}$ clusters at $y$. If $\{x_{n_k}\} \in C$ for infinitely many $k$, $\{x_{n_k}\}$ clusters at some $x \in C$, implying by the co-convergence of $U$ that $x = y$. Hence $U$ is contra-convergent and by Theorem 1, $X$ is metrizable.

## References


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[4] Sabella, R. R., Spaces in Which Compact Sets have Countable Local Bases, Proc. Amer. Math. Soc. (to appear).



(Paper 79B1&2–422)