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A Condition for the Diagonalizability of a Partitioned Matrix

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When U and V are diagonalizable matrices the diagonalizability of

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}$$

is equivalent to the solvability in X of

$$UX - XV = N.$$

A corollary and simple generalization are given.

Key Words: Diagonalizable matrix; partitioned matrix

A square complex matrix A is termed "diagonalizable" if and only if A is similar to a diagonal matrix; that is, if and only if there exists a nonsingular matrix S such that $S^{-1}AS$ is diagonal. Our purpose is to prove the following necessary and sufficient condition for the diagonalizability of a partitioned matrix. We shall denote the class of n by n complex matrices by $M_n(C)$.

THEOREM 1. *Suppose* $U, V \epsilon M_n(C)$ *are diagonalizable. Then*

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \text{ is similar to } \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$$

if and only if there is an $X \epsilon M_n(C)$ *such that* $UX - XV = N$.

PROOF: Suppose $S, T \epsilon M_n(C)$ are invertible and are such that

$$SUS^{-1} = D \text{ and } T^{-1}VT = E$$

are diagonal. Then

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \text{ is similar to } \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$$

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if and only if

$$\begin{bmatrix} D & SNT \\ 0 & E \end{bmatrix} \quad \text{is similar to} \quad \begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix},$$

and

$$UX - XV = N$$

if and only if

$$D(SXT) - (SXT)E = SNT.$$

Thus, it suffices to assume from the outset that U and V are diagonal matrices.

Now suppose, first of all, that

$$U = \operatorname{diag}\{u_1, \ldots, u_n\},$$

$$V = \operatorname{diag}\{v_1, \ldots, v_n\},$$

and that

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \quad \text{is similar to} \quad \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix},$$

where $N = (n_{ij}) \epsilon M_n(C)$. We will show that if $u_i = v_j$, then $n_{ij} = 0$. It follows that if $X = (x_{ij})$ is defined by $x_{ij} = \dfrac{n_{ij}}{u_i - v_j}$ for $u_i \neq v_j$, and x_{ij} arbitrary otherwise, then $UX - XV = N$.

Denote by E_{ij} the n by n matrix all of whose entries are 0 except for a 1 in the i,j position. Then

$$\begin{bmatrix} I & tE_{ij} \\ 0 & I \end{bmatrix} \begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \begin{bmatrix} I & -tE_{ij} \\ 0 & I \end{bmatrix} = \begin{bmatrix} U & N + t(v_j - u_i)E_{ij} \\ 0 & V \end{bmatrix}.$$

Because of this similarity we may assume without loss of generality that $n_{ij} = 0$ whenever $v_j \neq u_i$. If $N = 0$, we are finished. If not, we shall reach a contradiction. Suppose $N \neq 0$. Then via a permutational similarity we may assume that

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}$$

is such that

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix},$$

$u_i = v_j = u$ for $i \leqslant k, j \leqslant \ell$,
$u_i \neq u$, $v_j \neq u$ for $i > k, j > \ell$, and
$N_1 \neq 0$ is k by ℓ. It is then clear that

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}$$

is permutationally similar to

$$\left[\begin{array}{cc|cc} uI & N_1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & uI & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & * & N_2 \\ & & 0 & * \end{array}\right]$$

Since

$$\begin{bmatrix} uI & N_1 \\ 0 & uI \end{bmatrix} \text{ is similar to } \begin{bmatrix} uI & 0 \\ 0 & uI \end{bmatrix}$$

only if $N_1 = 0$, the assumption that $N \neq 0$ contradicts our original supposition that

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \text{ is similar to } \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}.$$

Thus our original supposition does imply that $n_{ij} = 0$ whenever $v_j = u_i$ which in turn implies that $UX - XV$ is solvable.

Finally suppose on the other hand that $X \epsilon M_n(C)$ is such that

$$UX - XV = N.$$

Then it is a simple computation that

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} U & -UX+N+XV \\ 0 & V \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix};$$

and the proof of the theorem is complete.

COROLLARY. *Suppose* $\mathrm{U}, \mathrm{V} \epsilon \mathrm{M_n(C)}$ *satisfy* $\mathrm{U}^p = \mathrm{V}^p = \mathrm{I}$, $p \epsilon \mathrm{I}^+$. *Then*

$$\begin{bmatrix} \mathrm{U} & \mathrm{N} \\ 0 & \mathrm{V} \end{bmatrix}^p = \mathrm{I}$$

if and only if there is an $\mathrm{X} \epsilon \mathrm{M_n(C)}$ *such that* $\mathrm{UX} - \mathrm{XV} = \mathrm{N}$.

PROOF: Since $U^p = V^p = I$, U and V are diagonalizable and the hypothesis of the theorem is satisfied. Now, if

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}^p = I,$$

then
$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix} \text{ is diagonalizable}$$

and thus similar to
$$\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}.$$

By the theorem, this implies $UX-XV=N$ is solvable. Conversely, if $UX-XV=N$, then

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}^p \quad \text{is similar to} \quad \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}^p$$

which is equal to I. thus

$$\begin{bmatrix} U & N \\ 0 & V \end{bmatrix}^p = I$$

which completes the proof of the corollary.

A straightforward generalization of Theorem 1 is as follows.

THEOREM 2. *Suppose each* $U_i \epsilon M_n(C)$, $i=1, \ldots, k$, *is diagonalizable. Then*

$$\begin{bmatrix} U_1 & & & & N_{ij} \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & U_k \end{bmatrix} \quad \textit{is similar to} \quad \begin{bmatrix} U_1 & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & U_k \end{bmatrix}$$

if and only if for each $i, j \leq k$ *there is an* $X \epsilon M_n(C)$ *(depending on* i *and* j*) such that* $U_iX - XU_j = N_{ij}$.

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