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The Field of Values and Spectra of Positive Definite Multiples*

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Suppose $0 \neq \lambda \epsilon C$ and $A \epsilon M_n(C)$. We show constructively that λ is an eigenvalue of HA for some $H^* = H > 0$ if and only if $\lambda = x^*Ax$ for some $x \epsilon C^n$. The characterization of H-stable matrices is then an easy corollary.

Key words: Eigenvalues; field of values; H-stable; positive definite matrix; spectrum.

Let $\sigma(A)$ denote the set of all eigenvalues of $A \epsilon M_n(C)$, the n by n complex matrices, and let

$$F(A) \equiv \{x^*Ax : x \epsilon C^n, x^*x = 1\},$$

the "field of values" of A.

In [1, 2, 4] [1] the H-stable matrices were characterized. (A matrix $A \epsilon M_n(C)$ is called H-stable if $\lambda \epsilon \sigma(HA)$ implies $\mathrm{Re}(\lambda) > 0$ for all $H^* = H > 0$.) The simplest version of the characterization is that A is H-stable if and only if A is nonsingular and $\lambda \epsilon F(A)$ implies $\mathrm{Re}(\lambda) > 0$ or $\lambda = 0$. In this note we give a simple characterization of those $\lambda \epsilon \sigma(HA)$ for some $H^* = H > 0$ and the theorem on H-stability, for example, is an easy corollary. The characterization is constructive in that a specific class of positive definite matrices H for which $\lambda \epsilon \sigma(HA)$ is produced. Let $\Sigma \equiv \{H \epsilon M_n(C) : H^* = H > 0\}$.

We first make two observations:

(1) $0 \epsilon \sigma(HA)$ for some $H \epsilon \Sigma$ if and only if $0 \epsilon \sigma(KA)$ for all $K \epsilon \Sigma$ if and only if $0 \epsilon \sigma(A)$;

(2) if $H = BB^*$, B nonsingular, then $\sigma(HA) = \sigma(B^*AB)$.

THEOREM 1: *Suppose* $0 \neq \lambda \epsilon \mathrm{C}$. *Then the following are equivalent for* $\mathrm{A} \epsilon \mathrm{M_n(C)}$:

(i) $\lambda = x^*Ax$, $x \epsilon C^n$;
(ii) $\lambda \epsilon \sigma(HA)$ *for some* $H \epsilon \Sigma$; *and*
(iii) $\lambda \epsilon \sigma(B^*AB)$ *for some nonsingular* $B \epsilon M_n(C)$.

PROOF: Since $H \epsilon \Sigma$ if and only if $H = BB^*$ for some nonsingular $B \epsilon M_n(C)$, it is clear from observation (2) that (ii) and (iii) are equivalent. To show that (iii) implies (i) suppose $\lambda \epsilon \sigma(B^*AB)$ with associated eigenvector y of length 1. Then $\lambda = y^*(B^*AB)y = (By)^*A(By) = x^*Ax$ for $x = By$. Conversely suppose $x^*Ax = \lambda$. Then $x \neq 0$ since $\lambda \neq 0$ and we let B_1 be any nonsingular matrix

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[1] Figures in brackets indicate the literature references at the end of this paper.

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in $M_n(C)$ whose first column is x. Then the 1, 1 entry of $B_1^* AB_1$ is λ. Let v be the vector formed by the remaining n-1 entries of the first row of $B_1^* AB_1$ and, in turn,$-z=\frac{1}{\lambda}v$. Then

$$B_2=\left[\begin{array}{c|c} 1 & z \\ \hline 0 & I \end{array}\right]$$

is nonsingular and $B_2^*(B_1^* AB_1)B_2$

$$=\left[\begin{array}{c|c} \lambda & 0 \\ \hline * & * \end{array}\right]$$

so that $\lambda\epsilon\sigma(B^*AB)$, where $B=B_1B_2$ is nonsingular which completes the proof.

Let R denote the real field. In the same manner as theorem 1 we may also prove:

THEOREM 1′: *Suppose* $0\neq\lambda\epsilon\mathrm{R}$. *Then the following are equivalent for* $\mathrm{A}\epsilon\mathrm{M}_n(\mathrm{R})$:

(i′) $\lambda=\mathrm{x}^{\mathrm{T}}\mathrm{Ax}$, $\mathrm{x}\epsilon\mathrm{R}^{\mathrm{n}}$.
(ii′) $\lambda\epsilon\sigma(\mathrm{HA})$ *for some* $\mathrm{H}\epsilon\Sigma\cap\mathrm{M}_n(\mathrm{R})$; *and*
(iii′) $\lambda\epsilon\sigma(\mathrm{B}^{\mathrm{T}}\mathrm{AB})$ *for some nonsingular* $\mathrm{B}\epsilon\mathrm{M}_{\mathrm{n}}(\mathrm{R})$.

We may exploit the above construction to give an explicit $H\epsilon\Sigma$ such that $0\neq\lambda\epsilon\sigma(HA)$ assuming that $A=(a_{ij})$ and $x=(x_1,\ .\ .\ .,x_n)$ such that $x^*Ax=\lambda$ are given. We assume without loss of generality that $x_1\neq 0$ and define

$$B_1\equiv\left[\begin{array}{c|c} x_1 & 0 \\ \hline x_2 & \\ \vdots & I \\ x_n & \end{array}\right],\quad B_2\equiv\left[\begin{array}{c|c} 1 & -\frac{1}{\lambda}\sum\limits_i \bar{x}_i a_{i2},\ .\ .\ .,\frac{1}{\lambda}\sum\limits_i \bar{x}_i a_{in} \\ \hline 0 & I \end{array}\right]$$

Then $B=B_1B_2$ is nonsingular and $H=B_1B_2B_2^*B_1^*\epsilon\Sigma$ with $\lambda\epsilon\sigma\ (HA)$.

Let $\mathscr{P}$ denote the open right half-plane and we have:

COROLLARY: $\mathrm{A}\epsilon\mathrm{M}_{\mathrm{n}}(\mathrm{C})$ *is* H-*stable if and only if* A *is nonsingular and* $\mathrm{F(A)}\subset\mathscr{P}\cup\{0\}$.

PROOF: Suppose A is H-stable. Then A is nonsingular and, since (ii) implies (i) in theorem 1, it follows that $F(A)\subset\mathscr{P}\cup\{0\}$. Suppose, alternatively, that $\lambda\epsilon F(A)$ and $\lambda\notin\mathscr{P}\cup\{0\}$. Then $Re(\lambda)\not>0$ and $\lambda\epsilon\sigma(HA)$ for some $H\epsilon\Sigma$. Thus A is not H-stable which completes the proof.

We close by mentioning a result parallel to theorem 1 which is proved elsewhere [3]. Let Γ denote an arbitrary open unbounded angular sector of the complex plane comprising less than 180 degrees. We then have

THEOREM 2: *The matrix* $\mathrm{A}\epsilon\mathrm{M}_{\mathrm{n}}(\mathrm{C})$ *may be written as* $\mathrm{A}=\mathrm{HB}$ *where* $\mathrm{H}\epsilon\Sigma$ *and* $\mathrm{F(B)}\subset\Gamma$ *if and only if* $\sigma(\mathrm{A})\subset\Gamma$.

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