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The Convex Hull of the Transposition Matrices

Lambert S. Joel

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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The convex hull of the n by n transposition matrices is characterized as the set of symmetric doubly stochastic matrices with trace $n-2$. A similar characterization (with trace $\geqslant n-2$) is given for the convex hull of the union of transposition matrices and the identity matrix.

Key words: Combinatorial analysis; convex set; linear inequalitites; permutations.

Many combinatorial optimization problems can be expressed as requiring the extremization of a linear function over some finite set S of points in a real N-dimensional space. To bring the theoretical and computational resources of linear programming to bear, it is necessary to characterize the convex hull $K(S)$ of S as the solution-set of a "nicely structured" family of linear inequalities and equations.

The outstanding example, arising in connection with the assignment problem of operations research, has $S=S_n$, the set of n by n permutation matrices (regarded as points in n^2 dimensional space). Here a well-known theorem[1] (Birkhoff-Hoffman-von Neumann-Wielandt et al.) identifies $K(S_n)$ as the set of all n by n doubly stochastic matrices $X=(x_{ij})$, i.e. matrices with nonnegative entries and with each row and column summing to 1. It is expected that a similar characterization of $K(C_n)$, where C_n is the set of all cyclic permutation matrices, would be valuable in connection with the traveling saleman problem, but no such characterization has been given as yet.

For a given n let $\Pi_c(1\leqslant c\leqslant n)$ denote the set of n by n permutation matrices for which the decomposition of the associated permutation into disjoint cycles contains exactly c cycles (including cycles of length one). Since $C_n=\Pi_1$, the remark ending the last paragraph suggests looking at the "other end" of the sequence $\{\Pi_c\}_{c=1}^{n}$. The situations for Π_n and Π_{n-1} are simple, and form the subject of this note. Clearly Π_n consists of the identity matrix I_n, so that $K(\Pi_n)=\{I_n\}$. We go on to characterize $K(\Pi_{n-1})$ as well as $K(\Pi_{n-1}\cup\Pi_n)=K(\Pi_{n-1}\cup\{I_n\})$. Note that Π_{n-1} consists of the $n(n-1)/2$ transposition matrices $T_{pq}(1\leqslant p<q\leqslant n)$ defined by

$$\begin{aligned} (T_{pq})_{ij}&=1 && \text{if } (i,j)=(p,q) \text{ or } (q,p),\\ (T_{pq})_{ii}&=1 && \text{for } i\neq p,\ q,\\ (T_{pq})_{ij}&=0 && \text{otherwise.} \end{aligned}$$

THEOREM 1. $K(\Pi_{n-1})$ *consists of all symmetric doubly stochastic matrices with trace* $n-2$.

THEOREM 2. $K(\Pi_{n-1})\cup\{I_n\})$ *consists of all symmetric doubly stochastic matrices with trace* $\geqslant n-2$.

To begin the proof, note that each member of Π_{n-1} (of $\Pi_{n-1}\cup\{I_n\}$) is a symmetric doubly stochastic matrix with trace $n-2$ (with trace $\geqslant n-2$). It readily follows that the same is true for each member of $K(\Pi_{n-1})$ (of $K(\Pi_{n-1}\cup\{I_n\})$).

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[1] Cf. Chap. 5: H. J. Ryser, Combinatorial Mathematics, Carus Mathematical Monograph No. 14, MAA, John Wiley & Sons 1963.

Next, let $X=(x_{ij})$ be any symmetric doubly stochastic matrix. Note that

$$\sum_{p}\sum_{q>p} X_{pq}=\frac{1}{2}\sum_{p,\,q\neq p} X_{pq}=\frac{1}{2}[n-\operatorname{tr}(X)]. \tag{1}$$

The matrix

$$Y=X-\sum_{p}\sum_{q>p} x_{pq}T_{pq}$$

is readily seen to be diagonal, with $(i,\,i)$ entry

$$y_{ii}=x_{ii}-\Sigma\{x_{pq}{:}p\neq i,\ q\neq i,\ q>p\}$$

$$=x_{ii}-\sum_{p<i}\left(\sum_{q>p}(x_{pq}-x_{pi})\right)-\sum_{p>i}\sum_{q>p} x_{pq}$$

$$=x_{ii}+\sum_{p<i} x_{pi}-\sum_{p\neq i}\sum_{q>p} x_{pq}$$

$$=\sum_{p\leq i} x_{pi}-\left[\sum_{p}\sum_{q>p} X_{pq}-\sum_{q>i} X_{iq}\right]$$

$$=\left[\sum_{p\neq i} x_{pi}+\sum_{q>i} x_{qi}\right]-\sum_{p}\sum_{q>p} x_{pq}.$$

Since column i of X sums to 1, the last result together with (1) yields

$$y_{ii}=1-\frac{1}{2}[n-\operatorname{tr}(X)]=\frac{1}{2}[\operatorname{tr}(X)-(n-2)]=\delta, \tag{2}$$

where the final equation defines δ.

If $\operatorname{tr}(X)=n-2$, then (2) yields $Y=0$ and (1) shows that $X=\Sigma x_{pq}T_{pq}$ lies in $K(\Pi_{n-1})$, completing the proof of Theorem 1. If $\operatorname{tr}(X)\geqslant n-2$, then $\delta\geqslant 0$; since (2) shows that $Y=\delta I_n$ and since (1) and (2) yield $\Sigma x_{pq}+\delta=1$, it follows that $X=\Sigma x_{pq}T_{pq}+\delta I_n$ lies in $K(\Pi_{n-1}\cup\{I_n\})$, completing the proof of Theorem 2.

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