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The Factorization of a Matrix as the Commutator of Two Matrices

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Let $P=I_p\dot{+}(-I_q)$, the direct sum of the $p\times p$ identity matrix and the negative of the $q\times q$ identity matrix. The following theorem is proved.

THEOREM: *If* X = cZ *where* Z *is a* 4×4 P-*orthogonal*, P-*skew-symmetric matrix and* |c| ≤ 2, *there exist matrices* A *and* B, *both of which are* P-*orthogonal and* P-*skew-symmetric, such that* X = AB − BA. Methods for obtaining certain matrices which satisfy $X=AB-BA$ are given. Methods are also given for determining pairs of anticommuting P-orthogonal, P-skew-symmetric matrices.

Key words: Anticommuting; commutator; factorization; matrix; orthogonal; skew-symmetric.

1. Introduction

Let $P=I_p\dot{+}(-I_q)$, the direct sum of the $p\times p$ identity matrix and the negative of the $q\times q$ identity matrix. Katz and Olkin [2] [1] define a real matrix A to be orthogonal with respect to P(P-orthogonal) if and only if

$$APA'=P \tag{1}$$

where A' is the transpose of A. Furthermore, they define B to be skew-symmetric with respect to P(P-skew-symmetric) if and only if BP is skew-symmetric in the ordinary sense.

The main result of this paper is concerned with matrices which are both P-orthogonal and P-skew-symmetric of order $n=4=p+q$. Smith [7] proved that such matrices exist in only two cases, $p=4$, $q=0$ and $p=q=2$. In the first case P-orthogonal and P-skew-symmetric reduce to orthogonal and skew-symmetric in the ordinary sense.

Pearl [4] and Smith [6] proved the following theorem in the cases $p=4$, $q=0$ and $p=q=2$ respectively.

THEOREM 1: *If the* 4×4 *matrices* A *and* B *are both* P-*orthogonal and* P-*skew-symmetric then their commutator*, [A, B] ≡ AB − BA, *is a scaler multiple of a* 4×4 P-*orthogonal*, P-*skew-symmetric matrix.*

The purpose of this paper is to prove a converse to Theorem 1. Shoda [5] proved that if X is a square matrix with zero trace having elements in an algebraically closed field then there exist matrices A and B such that $X=AB-BA$. Albert and Muckenhoupt [1] removed the restriction that the field be algebraically closed. However, both the method of Shoda and the method of Albert and Muckenhoupt give a singular matrix B. The main result of this paper is:

THEOREM 2: *If* X = cZ *where* Z *is a* 4×4 P-*orthogonal*, P-*skew-symmetric matrix and* |c| ≤ 2, *there exist matrices* A *and* B, *both of which are* P-*orthogonal and* P-*skew-symmetric, such that* X = AB − BA.

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[1] Figures in brackets indicate the literature references at the end of this paper.

Methods for obtaining certain matrices which satisfy $X=AB-BA$ are given. Methods are also given for determining pairs of anticommuting P-orthogonal, P-skew-symmetric matrices.

2. Anticommuting Matrices

In examining the structure of P-orthogonal, P-skew-symmetric matrices in the case $p=4$, $q=0$, Pearl [4] shows that any such matrix has exactly one of the following forms:

$$\text{(i)}\ \alpha_1 R_1+\alpha_2 R_2+\alpha_3 R_3, \qquad \alpha_1^2+\alpha_2^2+\alpha_3^2=1$$

$$\text{(ii)}\ \alpha_1 S_1+\alpha_2 S_2+\alpha_3 S_3, \qquad \alpha_1^2+\alpha_2^2+\alpha_3^2=1$$

where the α_i are real scalers and the R_i and S_i are the first and second regular representations respectively of the real quaternions [3].

Similarly, in the case $p=q=2$, Smith [6] shows that any such matrix has exactly one of the following forms:

$$\text{(iii)}\ \alpha_1 R_1 P+\alpha_2 S_2 P+\alpha_3 S_3 P, \qquad \alpha_2^2+\alpha_3^2-\alpha_1^2=1$$

$$\text{(iv)}\ \alpha_1 S_1 P+\alpha_2 R_2 P+\alpha_3 R_3 P, \qquad \alpha_1^2-\alpha_2^2-\alpha_3^2=-1$$

where $P=I_2 \dot{+} (-I_2)$.

A further examination of these papers leads to

THEOREM 3: *If* Z *is a* 4×4 P-*orthogonal*, P-*skew-symmetric matrix there exists a* 4×4 P-*orthogonal*, P-*skew-symmetric matrix* B *such that* ZB $=-$ BZ.

PROOF: There are four cases to consider.

Case 1, $Z=\alpha_1 R_1+\alpha_2 R_2+\alpha_3 R_3$. If $\alpha_3\neq 0$, choose arbitrary β_1', β_2' and set

$$\beta_3'=-\frac{1}{\alpha_3}(\alpha_1\beta_1'+\alpha_2\beta_2').$$

Let $x=\beta_1'^2+\beta_2'^2+\beta_3'^2$ and set $\beta_i=\dfrac{\beta_i'}{\sqrt{x}}$, $i=1,2,3$. If $\alpha_3=0$, let $\beta_1=\beta_2=0$ and $\beta_3=1$. Clearly, in either situation

$$\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=0 \tag{2}$$

and

$$\beta_1^2+\beta_2^2+\beta_3^2=1. \tag{3}$$

Letting $B=\beta_1 R_1+\beta_2 R_2+\beta_3 R_3$, by (3) B is P-orthogonal, P-skew-symmetric and by (2)

$$ZB=-BZ.$$

Case 2, $Z=\alpha_1 S_1+\alpha_2 S_2+\alpha_3 S_3$. Choose β_i, $i=1,2,3$ as in Case 1 and let

$$B=\beta_1 S_1+\beta_2 S_2+\beta_3 S_3.$$

Case 3, $Z=\alpha_1 R_1 P+\alpha_2 S_2 P+\alpha_3 S_3 P$. The matrix $B=\beta_1 R_1 P+\beta_2 S_2 P+\beta_3 S_3 P$ will be P-orthogonal, P-skew-symmetric if

$$\beta_2^2+\beta_3^2-\beta_1^2=1 \tag{4}$$

and $ZB=-BZ$ if

$$\alpha_2\,\beta_2+\alpha_3\,\beta_3-\alpha_1\,\beta_1=0. \tag{5}$$

If $\alpha_1+\alpha_2\neq 0$, set $\beta_1=\dfrac{\alpha_3}{\alpha_1+\alpha_2}$, $\beta_2=\dfrac{-\alpha_3}{\alpha_1+\alpha_2}$, and $\beta_3=1$. Clearly, (4) and (5) are satisfied. If $\alpha_1=\alpha_2=0$, set $\beta_1=\beta_3=0$, $\beta_2=1$. Again (4) and (5) are satisfied. If $\alpha_1=-\alpha_2\neq 0$, since clearly $\alpha_3=\pm 1$, set $x=\dfrac{1}{1+\alpha_1^2}$ and let $\beta_1=0$, $\beta_3=\alpha_3\,\alpha_1\sqrt{x}$, $\beta_2=\sqrt{x}$. Again (4) and (5) are satisfied.

Case 4, $Z=\alpha_1\,S_1\,P+\alpha_2\,R_2\,P+\alpha_3\,R_3\,P$. Let $B=\beta_1\,S_1\,P+\beta_2\,R_2\,P+\beta_3\,R_3\,P$ where the β_i are chosen as in case 3.

3. Proof of Theorem 2

In order to prove Theorem 2 it is convenient to first prove the following lemmas.

LEMMA 1: (i) *If* B *is* P-*skew-symmetric then* $B'=-PBP$.

(ii) *If* B *is* P-*skew-symmetric and* P-*orthogonal than* $B^2=-I$.

LEMMA 2: *If* Z *is a* P-*orthogonal,* P-*skew-symmetric matrix and* $|c|\leqslant 2$ *then*

$$Y=\frac{\sqrt{4-c^2}}{2}\,I+\frac{c}{2}\,Z \text{ is P-orthogonal.}$$

PROOF: By direct computation,

$$YPY'=\left(\frac{\sqrt{4-c^2}}{2}\,I+\frac{c}{2}\,Z\right)P\left(\frac{\sqrt{4-c^2}}{2}\,I+\frac{c}{2}\,Z'\right)$$

$$=\frac{4-c^2}{4}\,P+\frac{c^2}{4}\,ZPZ'+\frac{\sqrt{4-c^2}}{2}\,(ZP+PZ').$$

However, $ZPZ'=P$ by (1) and by Lemma 1

$$ZP+PZ'=ZP+P(-PZP)=ZP-ZP=0.$$

Thus $YPY'=\dfrac{4-c^2}{4}\,P+\dfrac{c^2}{4}\,P+O=P$ and by (1) Y is P-orthogonal.

LEMMA 3: *If* Z *is* P-*orthogonal,* P-*skew-symmetric and* $|c|\leqslant 2$, *and if* B *is* P-*orthogonal,* P-*skew-symmetric such that* $ZB=-BZ$, *then* $A=\left(\dfrac{\sqrt{4-c^2}}{2}\,I+\dfrac{c}{2}\,Z\right)PB'P$ *satisfies* $[A,B]=cZ$.

PROOF: $AB=(YPB'P)\,B=(YP)\,(B'PB)=(YP)\,P=Y$

$$=\frac{\sqrt{4-c^2}}{2}\,I+\frac{c}{2}\,Z$$

$$BA=B\,(YPB'P)=\frac{\sqrt{4-c^2}}{2}\,BPB'P+\frac{c}{2}\,BZPB'P$$

$$=\frac{\sqrt{4-c^2}}{2}\,I+\frac{c}{2}\,BZPB'P$$

$$=\frac{\sqrt{4-c^2}}{2}\,I-\frac{c}{2}\,ZBPB'P$$

$$=\frac{\sqrt{4-c^2}}{2} I-\frac{c}{2} Z.$$

Thus $[A,B]=AB-BA=cZ$.

COROLLARY: *The matrix* A *defined in Lemma 3 is* P-*orthogonal*, P-*skew-symmetric.*

PROOF: By Lemmas 1 and 2 A is the product of two P-orthogonal matrices. Hence A is P-orthogonal. Also

$$A=\left(\frac{\sqrt{4-c^2}}{2} I+\frac{c}{2} Z\right) PB'P$$

$$=\frac{\sqrt{4-c^2}}{2} PB'P+\frac{c}{2} ZPB'P.$$

By Lemma 1, $\frac{\sqrt{4-c^2}}{2} PB'P=-\frac{\sqrt{4-c^2}}{2} B$ which is P-skew-symmetric. Furthermore

$$\left(\frac{c}{2} ZPB'P\right)'=\frac{c}{2} PBPZ'$$

$$=-\frac{c}{2} B'Z' \qquad \text{by Lemma 1}$$

$$=\frac{c}{2} Z'B' \qquad \text{since } ZB=-BZ$$

$$=-\frac{c}{2} PZPB' \qquad \text{by Lemma 1}$$

$$=-\frac{c}{2} P\left(ZPB'P\right)P.$$

Thus A is the sum of two P-skew-symmetric matrices and hence A is P-skew-symmetric.

In the 4×4 case, the existence of the matrix B is given by Theorem 3. Thus Theorem 3, Lemma 3, and the Corollary complete the proof of Theorem 2.

4. Conclusion

Theorem 2 provides a converse to the theorems of Pearl [4] and Smith [6]. While Theorem 2 is restricted to the 4×4 case, the results of section 3 refer to the general $n\times n$ case. Smith [8] has generalized Theorem 1 to the $n\times n$ case. Perhaps the results of section 3 can be applied to find a converse of that result.

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