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A Sufficient Condition for Matrix Stability*

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An n by n complex matrix A is said to be positive stable if Re $(\lambda) > 0$ for each eigenvalue λ of A. If A satisfies both of the following two conditions, then A is positive stable: (1) for each $k=1, \ldots, n$, the real part of the sum of the k by k principal minors of A is positive; and (2) the minimum of the real parts of the eigenvalues of A is itself an eigenvalue of A. Special cases include hermitian positive definite matrices and M-matrices.

Key words: M-matrix; stable matrix; principal minors.

Suppose $A=(a_{ij})\epsilon M_n(C)$, the n by n complex matrices. Then let $P_k(A)$ denote the sum of the k by k principal minors of A. If Re $(\lambda) > 0$ for each eigenvalue λ of A, then A is called (positive) *stable*, and if Re $[P_k(A)] > 0$, $k=1, \ldots, n$, then A is called *prestable* [2].[1] This note was suggested by the following question which arose in the stability analysis of an economic equilibrium [3]. Suppose $A \epsilon M_n(R)$, the real n by n matrices, has the sign pattern (M):

$$a_{ii} > 0, \qquad i=1, \ldots, n$$

and

$$a_{ij} \leq 0, \qquad i \neq j, \; i, j=1, \ldots, n.$$

If A *is prestable, does it then follow that* A *is stable?* Unfortunately, this is not one of the 13 equivalent conditions given in [1] for A to be an M-matrix. An affirmative answer to this question is a corollary to what we shall prove.

LEMMA 1: *If* $A \epsilon M_n(C)$ *is prestable, then* A *has no nonpositive real eigenvalues.*

PROOF: Suppose A is prestable and that $-r$, $r \geq 0$, is a nonpositive real eigenvalue of A. But then $A-(-r)I=A+rI$ is singular which means det $(A+rI)=0$. However,

$$\det (A+rI)=r^n+\sum_{k=1}^{n} P_k(A) r^{(n-k)}$$

so that $\text{Re}[\det(A+rI)] > 0$, a contradiction. This means there is no such r and completes the proof.

When the minimum of the real parts of the eigenvalues of $A \epsilon M_n(C)$ is itself an eigenvalue of A, we shall say that A has the *property* (*).

THEOREM: *Suppose* $A \epsilon M_n(C)$ *has property* (*). *If* A *is prestable, then* A *is stable.*

PROOF: Because of lemma 1 and property (*), the minimum of the real parts of the eigenvalues of A is positive. This means A is stable.

Since it is clear that the well known class of hermitian matrices satisfies property (*), it follows that:

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[1] Figures in brackets indicate the literature references at the end of this paper.

Corollary 1: *If* $A \epsilon M_n(C)$ *is hermitian, then* A *prestable implies* A *stable.*
Another well known class of matrices which has property (*) are those of sign pattern (M).

Lemma 2: *If* $A \epsilon M_n(R)$ *has sign pattern* (M), *then* A *has property* (*).

Proof: Suppose that m is a real number greater than each of the a_{ii}. Then A may be written

$$A = mI - P$$

where P is an entry-wise nonnegative matrix. Because of the Perron-Frobenius theorem, P has a dominant positive eigenvalue r such that $r \geqslant |\lambda|$, and thus $r \geqslant \mathrm{Re}\,(\lambda)$, for each eigenvalue λ of P. It then follows that $m - r$ is an eigenvalue of A and is the smallest of the real parts of the eigenvalues of A, so that A has property (*).

Corollary 2: *If* $A \epsilon M_n(R)$ *has sign pattern* (M), *then* A *prestable implies* A *stable.*

Proof: This follows from Lemma 2 and the theorem.

We close with a question of interest which arises from the preceding remarks:
What are necessary and sufficient conditions on A *such that* A *satisfy property* (*)?

References

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[3] Quirk, J., Private communication, December 1973.

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