JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematical Sciences
Vol. 78B, No. 2, April–June 1974

A New Proof of Pick's Theorem*

Steven Minsker**

(March 18, 1974)

Pick's theorem states that, if f is a univalent analytic function on the open unit disk with $f(0)=0$ and $f'(0)=1$, and $|f|\leq M$, then $\left|\frac{f''(0)}{2}\right|\leq 2\left(1-\frac{1}{M}\right)$. A new proof of this result is given, and a comparison with the usual proof is made.

Key words: Bieberbach's theorem; coefficient estimate; univalent analytic function.

Let $\mathcal{F}$ denote the collection of all univalent analytic functions $f:\Delta(0;\,1)\rightarrow \mathbb{C}$ with $f(0)=0$ and $f'(0)=1$. Such functions have power series expansions of the form

$$f(z)=z+a_2z^2+a_3z^3+\ .\ .\ .\ .$$

Bieberbach's famous theorem states that $|a_2|\leq 2$ for all $f\in\mathcal{F}$, with equality holding only for the functions $g_\beta(z)=\frac{z}{(1+e^{i\beta}z)^2}$, β real. The theorem that will interest us here is as follows:

THEOREM (*Pick*, [2])[1]: *If* $f\in\mathcal{F}$ *and* $|f|\leq M$, *then* $|a_2|\leq 2\left(1-\frac{1}{M}\right)$.

The usual proof consists of an application of Bieberbach's theorem to the functions

$$f_\beta(z)=\frac{f(z)}{(1+e^{i\beta}f(z)/M)^2},$$

β real. (See, for instance, [1], p. 224, ex. 4.) We offer the following proof which, although it also uses Bieberbach's result, is considerably different.

PROOF: Let $\phi(z)=\frac{f(z)}{M}$. Then ϕ sends the open unit disk into itself. Let $\phi_1=\phi$ and inductively define $\phi_n=\phi\circ\phi_{n-1}$. If $\phi_n(z)=A_{n,1}\ z+A_{n,2}\ z^2+\ .\ .\ .$, it is clear that $A_{1,1}=\frac{1}{M}$, $A_{1,2}=\frac{a_2}{M}$, $A_{n,1}=A_{1,1}\,A_{n-1,1}$, and $A_{n,2}=A_{1,1}\,A_{n-1,2}+A_{1,2}\,A^2_{n-1,1}$. It follows that $A_{n,1}=A^n_{1,1}$ and

$$A_{n,2}=A_{1,2}A^{n-1}_{1,1}(1+A_{1,1}+A^2_{1,1}+\ .\ .\ .\ +A^{n-1}_{1,1}).$$

Now $\phi_n/A_{n,1}\in\mathcal{F}$ for each n, so Bieberbach's theorem implies that $|A_{n,2}/A_{n,1}|\leq 2$, or

$$|A_{1,2}/A_{1,1}|\,(1+A_{1,1}+A^2_{1,1}+\ .\ .\ .\ +A^{n-1}_{1,1})\leq 2.$$

Consider two cases:

(1) $A_{1,1}=1$ (that is, $M=1$). Then $|A_{1,2}|n\leq 2$ for all n, so $A_{1,2}=0$ and $a_2=0$. Thus $|a_2|\leq 2\left(1-\frac{1}{M}\right)$ trivially.

AMS Subject Classification: 30A34.

*An invited paper. This note is based on the author's Ph. D. thesis, M.I.T., June 1972.

**Present address: Department of Mathematics, University of Pittsburgh, Pa. 15260.

[1] Figures in brackets indicate the literature references at the end of this paper.

536-595 O - 74 - 3

(2) $A_{1,1}<1$ (that is, $M>1$). Then, letting $n\rightarrow\infty$, we get $|A_{1,2}/A_{1,1}|\frac{1}{1-A_{1,1}}\leqslant 2$, so $|a_2|\leqslant 2\left(1-\frac{1}{M}\right)$. Since the case $M<1$ cannot occur (as is seen by applying Schwarz's lemma to f/M), the proof is complete.

REMARK: The two proofs are related in the following fashion. Suppose that $M>1$ and that equality holds in Pick's theorem, that is, $a_2=-2\left(1-\frac{1}{M}\right)e^{i\beta_0}$, β_0 real. Then $A_{n,2}/A_{n,1}\rightarrow -2e^{i\beta_0}$. Now it is a consequence of the Koebe distortion theorem and Hurwitz's theorem that $\mathscr{F}$ is compact for the topology of uniform convergence on compact subsets of $\Delta(0;1)$. So any convergent subsequence of $\{\phi_n/A_{n,1}\}$ must converge to a function in $\mathscr{F}$ with second coefficient $-2e^{i\beta_0}$, and hence must converge to g_{β_0}. It follows that $\phi_n/A_{n,1}\rightarrow g_{\beta_0}$. But then $g_{\beta_0}\circ\phi=\lim\ (\phi_n\circ\phi)/A_{n,1}=\frac{1}{M}\lim \phi_{n+1}/A_{n+1,1}=\frac{1}{M}g_{\beta_0}$. So $g_{\beta_0}=M(g_{\beta_0}\circ\phi)=f_{\beta_0}$. Thus, we have shown that, if $M>1$ and equality occurs in Pick's theorem, the sequence $\{\phi_n/A_{n,1}\}$ converges to f_{β_0} for some β_0. And we have shown that, for equality to occur, the function f must satisfy $f_{\beta_0}=g_{\beta_0}$ for some β_0 (this is also true for $M=1$, for then $f(z)=z$ by Schwarz's lemma and $f_\beta=g_\beta$ for all β). This condition turns out to be sufficient also, so equality holds in Pick's theorem only for the functions

$$h_\beta(z)=e^{i\beta}M\,\frac{z+e^{i\beta}-\sqrt{(z+e^{i\beta})^2-4e^{i\beta}z/M}}{z+e^{i\beta}+\sqrt{(z+e^{i\beta})^2-4e^{i\beta}z/M}},\ \beta \text{ real.}$$

References

[1] Nehari, Z., Conformal Mapping (McGraw-Hill, 1952).
[2] Pick, G., Über die konforme Abbildung eines Kreises auf ein schlichtes und zugleich beschränktes Gebiet, Sitzgber. Kaiserl. Akad. Wiss. Wien, Math. – naturwiss. Kl. Abt. IIa 126 (1917), 247–263.

(Paper 78B2–404)