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Rational Equivalence of Unimodular Circulants*

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We answer a question of M. Newman by providing all unimodular positive circulants are rationally equivalent to the identity.

Key words: Circulant, totally positive unit.

Let P be the $n \times n$ matrix satisfying $P_{12} = \ldots = P_{n-1,\,n} = P_{n,\,1} = 1$ and all other entries 0. The elements of the group ring $Z[P]$ are called integral circulants. Let G be the group in $Z[P]$ consisting of the positive definite symmetric unimodular elements. M. Newman [2, p. 198] asks which members of G are rationally equivalent to I_n. The answer to this question is in fact an easy consequence of the Hasse norm theorem [1, p. 186].

THEOREM: *All members of* G *are rationally equivalent to* I_n.

PROOF: Let $A \epsilon G$. As noted in [2, p. 198], we must show that any eigenvalue λ of A is of the form $\alpha\bar{\alpha}$ for some α in Q_n, the nth cyclotomic field. Let K be the real subfield of index 2 in Q_n. We must show λ is a norm from Q_n to K. If $n = 2p^n$, p a prime, then p is fully ramified in Q_n. In any other case, Q_n is unramified over K at all finite primes. So at most one finite prime ramifies from K to Q_n.

Now λ is totally positive, so λ is a norm at all Archimedean localizations. And λ is a unit; thus λ is a norm at all finite localizations, except, possibly, one. By the product formula, λ is a norm everywhere, and hence, by the Hasse norm theorem, λ is a global norm.

References

[1] Hasse, H., Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper (1930).
[2] Newman, M., Integral Matrices (Academic Press, 1972).
[3] O'Meara, Introduction to Quadratic Forms (Academic Press, 1963).

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