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The Smith Normal Form of a Partitioned Matrix

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It is shown that if

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ 0 & M_{22} & \cdots & M_{2t} \\ & & \ddots & \\ 0 & 0 & \cdots & M_{tt} \end{bmatrix}$$

is a matrix over a principal ideal ring R such that the matrices M_{ii} are square and have pairwise relatively prime determinants, then the Smith normal form of M is the same as the Smith normal form of

$M_{11} \dot{+} M_{22} \dot{+} \cdots \dot{+} M_{tt}$.

Key words: Elementary divisors; invariant factors; partitioned matrices; Smith normal form.

Let R be a principal ideal ring. Let A be an $r \times r$ matrix over R, B an $s \times s$ matrix over R. It is well known that the elementary divisors of $A \dot{+} B$ are the elementary divisors of A together with the elementary divisors of B, which allows us to reconstruct the Smith Normal Form (hereafter abbreviated S.N.F.) of $A \dot{+} B$ from the invariant factors of A and of B (see [1],[1] for example). There is a noteworthy instance which merits special attention: namely, when the determinants of A and B are relatively prime. This note is devoted to this case.

We let $S(M)$ denote the S.N.F. of any matrix M over R, and I_n denote the identity matrix of order n. I will denote an identity matrix of unspecified order.

We first prove

THEOREM 1: *Suppose that* $(det(\mathrm{A}), det(\mathrm{B}))=1$, *and that*

$$\mathrm{S(A)} = diag(\alpha_1, \alpha_2, \ldots, \alpha_r),$$

$$\mathrm{S(B)} = diag(\beta_1, \beta_2, \ldots, \beta_s),$$

so that $\alpha_1, \alpha_2, \ldots, \alpha_r$ *are the invariant factors of* A, $\beta_1, \beta_2, \ldots, \beta_s$ *the invariant factors of* B; *and assume for definiteness that* $r \leq s$. *Then*

(1) $\mathrm{S(A \dot{+} B)} = \mathrm{I_r} \dot{+} diag(\beta_1, \beta_2, \ldots, \beta_{s-r}) \dot{+} diag(\alpha_1\beta_{s-r+1}, \alpha_2\beta_{s-r+2}, \ldots, \alpha_s\beta_r)$.

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[1] Figures in brackets indicate the literature references at the end of this paper.

PROOF: A moment's consideration shows that the expression given in (1) for $S(A\dot{+}B)$ is just $S(I_s\dot{+}A)\,S(I_r\dot{+}B)$. Now

$$A\dot{+}B=\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}=\begin{bmatrix} A & 0 \\ 0 & I_s \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & B \end{bmatrix}$$

$$=(A\dot{+}I_s)\,(I_r\dot{+}B).$$

But $A\dot{+}I_s$ and $I_r\dot{+}B$ have relatively prime determinants, and it is known that if M, N are matrices over R of the same size such that $(\det(M), \det(N))=1$, then $S(MN)=S(M)S(N)$ (see [2]). It follows that

$$S(A\dot{+}B)=S(A\dot{+}I_s)\,S\,(I_r\dot{+}B)$$

$$=S(I_s\dot{+}A)\,S\,(I_r\dot{+}B).$$

This concludes the proof.

Now let T be any $r\times s$ matrix over R. Then provided that $(\det(A), \det(B))=1$, the next result shows that T plays no part in determining the S.N.F. of

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}.$$

Specifically, we prove

THEOREM 2: *Suppose that* $(\det(A), \det(B))=1$. *Then the* S.N.F. *of*

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$$

is the same as the S.N.F. *of*

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

PROOF: Let A^{adj} be the adjoint of A, B^{adj} the adjoint of B, so that A^{adj}, B^{adj} are matrices over R satisfying

$$AA^{\mathrm{adj}}=A^{\mathrm{adj}}A=\det(A)\cdot I_r,$$

$$BB^{\mathrm{adj}}=B^{\mathrm{adj}}B=\det(B)\cdot I_s.$$

Since $(\det(A), \det(B))=1$, elements α, β of R exist such that

$$\alpha\det(A)+\beta\det(B)=1.$$

Now consider the equation

$$\begin{bmatrix} I_r & X \\ 0 & I_s \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I_r & Y \\ 0 & I_s \end{bmatrix}=\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}. \tag{2}$$

Then (2) holds if and only if

$$T=AY+XB. \tag{3}$$

But (3) may be satisfied by choosing

$$X=\beta TB^{\mathrm{adj}},\ Y=\alpha A^{\mathrm{adj}}T.$$

Thus (2) has a solution in matrices X, Y over R, and it follows that $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are equivalent, and hence have the same S.N.F. This concludes the proof.

We remark that because of Theorem 1, the S.N.F. of $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$ is completely determined by the invariant factors of A and the invariant factors of B, when $(\det(A), \det(B))=1$.

This result is definitely false if the determinant condition is removed. For example, the S.N.F. of $\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$ is $\begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix}$, but the S.N.F. of $\begin{bmatrix} 4 & 1 \\ 0 & 6 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 24 \end{bmatrix}$.

We note in passing that if

$$S(A)=\mathrm{diag}\ (\alpha_1, \alpha_2, \ldots, \alpha_r),$$

then

$$S(A\dot{+}A\dot{+}\ldots\dot{+}A)=\alpha_1 I_k\dot{+}\alpha_2 I_k\dot{+}\ldots\dot{+}\alpha_r I_k,$$

where there are k replicas of A in the direct sum.

Theorem 2 may be generalized as follows:

THEOREM 3: *Let* M *be a matrix over* R, *and suppose that* M *may be partitioned as*

$$\mathrm{M}=\begin{bmatrix} \mathrm{M}_{11} & \mathrm{M}_{12} & \ldots & \mathrm{M}_{1t} \\ 0 & \mathrm{M}_{22} & \ldots & \mathrm{M}_{2t} \\ & \ldots & & \\ 0 & 0 & \ldots & \mathrm{M}_{tt} \end{bmatrix},$$

where the matrices M_{ii} *are square and have pairwise relatively prime determinants. Then the* S.N.F. *of* M *is determined by the invariant factors of the* M_{ii};

$$\mathrm{S}(\mathrm{M})=\mathrm{S}(\mathrm{M}_{11}\dot{+}\mathrm{M}_{22}\dot{+}\ldots\dot{+}\mathrm{M}_{tt}).$$

PROOF: Put

$$A=M_{11},$$

$$T=[M_{12}, \ldots, M_{1t}],$$

$$B=\begin{bmatrix} M_{22} & \ldots & M_{2t} \\ & \ldots & \\ 0 & \ldots & M_{tt} \end{bmatrix}.$$

Then

$$M=\begin{bmatrix} A & T \\ 0 & B \end{bmatrix},$$

and $(\det(A), \det(B))=1$. By Theorem 2 and the result on the multiplicativity of the S.N.F.,

$$S(M)=S\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)$$

$$=S\left(\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}\right) S\left(\begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}\right),$$

since $(\det(A), \det(B))=1$. Repeating this procedure with the matrix B, we ultimately obtain

$$S(M)=S(M_{11}\dotplus I)S(I\dotplus M_{22}\dotplus I)\ldots S(I\dotplus M_{tt})$$

$$=S(M_{11}\dotplus M_{22}\dotplus\ldots\dotplus M_{tt}).$$

This concludes the proof.

References

[1] Newman, Morris, Integral Matrices, Academic Press, New York (1972).
[2] ———, On the Smith normal form, J. Res. Nat. Bur. Stand. (U.S.), 75B (Math. Sci.), Nos. 1&2, 81–84 (Jan.–June 1971).

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