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On Characters of Subgroups *

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Let H be a subgroup of G. Let χ be an irreducible character of H. Let χ^G be the character of G induced by χ. The irreducibility of χ^G is discussed. In particular, if H is normal in G, then χ^G is irreducible if and only if χ cannot be extended to any subgroup of G which properly contains H.

These results have application to the determination of irreducibility of a class of representations of the full linear groups.

Key words: Frobenius Reciprocity Theorem.

I. First Introduction

Let H be a subgroup of G. Let χ be an irreducible complex character of H. In the course of the author's study of a class of representations of the full linear group, the following criterion arose:

Condition 1. There is an irreducible character λ of G such that $(\lambda, \chi)_H=1$, and $(\xi, \chi)_H=0$, for every irreducible character ξ of G different from λ. (Here, of course,

$$(\xi, \chi)_H=\frac{1}{o(H)} \sum_{h\epsilon H} \xi(h)\, \chi(h^{-1}).$$

It turns out that when $G=S_m$, the symmetric group, Condition 1 is equivalent to the irreducibility of a certain representation of the full linear group [3].[1] The main purpose of this (essentially expository) note is to investigate character theoretic statements which are related to Condition 1.

II. Second Introduction

Let H be a subgroup of G and let χ be an irreducible character of H. Can we obtain from this situation any information about the irreducible characters of G? It would be most pleasant, for example, if χ could be extended to a character of G. But, this is not always possible.

One general method to obtain a character of G from χ goes as follows: Define χ^* on G by $\chi^*(h)=\chi(h)$ for $h \epsilon H$, and $\chi^*(g)=0$ for $g \epsilon G\setminus H$. then

$$\chi^G(g)=\frac{1}{o(H)} \sum_{f\epsilon G} \chi^*(f^{-1}g f), \qquad g \epsilon G,$$

turns out to be a character of G whose degree is $\chi(id)\ [G:H]$. It is called the character of G *induced* by χ [1, 2].

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[1] Figures in brackets indicate the literature references at the end of this paper.

Now, of course, we would like to know something about χ^G. For example, is it an irreducible character? In general, the answer is no. We are indebted to Frobenius for the following very useful result:

(Frobenius Reciprocity) Theorem: Let H be a subgroup G. Let χ and λ be characters of H and G respectively. Then

$$(\chi, \lambda)_H = (\chi^G, \lambda)_G.$$

As we shall see, the irreducibility of χ^G is related to the extendability of χ.

III. Results

Suppose $g \epsilon G$. We let χ^g denote the character of gHg^{-1} defined by

$$\chi^g(gHg^{-1}) = \chi(h), \qquad h \epsilon H.$$

THEOREM 1: Let H be a subgroup of G. Let χ be an irreducible character of H. The following are equivalent

a. Condition 1.
b. χ^G is irreducible (in fact $\chi^G = \lambda$).
c. For all $g \in G \setminus H$, χ^g and χ are different characters of $H \cap gHg^{-1}$.

THEOREM 2: If χ^G is irreducible, then χ cannot be extended to any subgroup of G which properly contains H.

Unfortunately, the converse of Theorem 2 is not true in general. For example, let $G = S_4$. Let H be the subgroup generated by $\{(14)(23), (1234)\}$. (Then H is the dihedral group D_4 of order 8.) Let χ be the irreducible character of H of degree 2. The only subgroup of G which properly contains H is G itself, and χ does not extend to G. The character χ^G, of degree 6, is the sum of the two inequivalent characters of G of degree 3.

When H is normal in G, however, the converse does hold.

THEOREM 3: Let H be a normal subgroup of G. Let χ be an irreducible character of H. If χ cannot be extended to any subgroup of G which properly contains H, then χ^G is irreducible.

In this connection, we point out a recent result of Roth [4, Theorem 3.1].

(Roth's) Theorem. Let ξ be a character of G of degree 1. Let $H = ker\ \xi = \{g \in G : \xi(g) = 1\}$. Suppose there exists an irreducible character λ of G such that $\lambda\xi = \lambda$. Then there exists an irreducible character χ on H such that $\chi^G = \lambda$.

IV. Proofs

We begin with Theorem 1. The equivalence of a and b is immediate from the Frobenius Reciprocity Theorem, i.e., $\chi^G = \lambda$ if and only if $(\lambda, \chi)_H = 1$, and $(\xi, \chi)_H = 0$, for every irreducible character ξ of G different from λ. The equivalence of b and c is Theorem (45.2)′ of [1].

The proof of Theorem 2 is equally straight forward. If χ^G is irreducible, then (by Theorem 1) for all $g \in G \setminus H$, χ^g and χ are different on

$$H \cap gHg^{-1} \subset \langle H, g \rangle,$$

the group generated by H and g. Thus, since characters are class functions, χ cannot be extended to $\langle H, g \rangle$.

We proceed to the proof of Theorem 3.

LEMMA: Let H be a normal subgroup of G. Let χ be an irreducible character on H. Then χ can be extended to $\langle H, g \rangle$ if and only if $\chi^g = \chi$.

PROOF: As above, necessity is clear. Suppose, then, that $\chi(g^{-1}hg)=\chi(h)$ for all $h \in H$. Let $h \rightarrow A(h)$ be an irreducible representation of H affording χ. Define

$$B(h) = A(g^{-1}hg), \qquad h \in H.$$

Then $h \rightarrow B(h)$ is a representation of H which affords χ. It follows that A and B are equivalent. Let U be nonsingular such that

$$B(h) = U^{-1}A(h)U, \qquad h \in H. \tag{1}$$

Now, let r be minimal such that $g^r \in H$. Observe

$$\begin{aligned} A(g^{-r})A(h)A(g^r) &= A(g^{-r}hg^r) \\ &= A(g^{-1}g^{-r+1}hg^{r-1}g) \\ &= B(g^{-r+1}hg^{r-1}) \\ &= U^{-1}A(g^{-r+1}hg^{r-1})U \\ &= \ldots \\ &= U^{-r}A(h)U^r. \end{aligned}$$

Thus, $A(g^r)U^{-r}$ commutes with $A(h)$ for all $h \in H$. It follows from Schur's Lemma that $A(g^r)U^{-r}$ is a scalar matrix S. We now replace U in (1) with U times any scalar rth root of S^{-1}, i.e., we may assume that $U^r = A(g^r)$.

Next, we define R on $<H, g>$ by

$$R(hg^k) = A(h)U^k$$

for all $h \in H$ and $k=0, 1, \ldots, r-1$. We claim R is a representation of $< H, g >$. Observe

$$R(h_1g^s)R(h_2g^t) = A(h_1)U^sA(h_2)U^t \tag{2}$$

and

$$R(h_1g^sh_2g^t) = R(h_1h_2'g^{s+t}) = A(h_1)A(h_2')U^{s+t}, \tag{3}$$

where $h_2' = g^sh_2g^{-s}$. To obtain equality between (2) and (3), it remains to show that

$$U^sA(h_2) = A(h_2')U^s.$$

But, this follows as above. This establishes our claim that R is a representation of $<H, g>$. Since the restriction of R to H is A, the character afforded by R extends χ. The proof of the lemma is complete.

Now, to complete the proof of Theorem 3, we appeal to the implication $c \rightarrow b$ of Theorem 1.[2] Since $gHg^{-1} = H$, this implication establishes that χ^G is irreducible if $\chi \neq \chi^g$ for all $g \in G \setminus H$, i.e., by the lemma, if χ cannot be extended to $< H,h >$ for all $g \in G \setminus H$.

COROLLARY: Let H be a normal subgroup of G. Suppose $[G:H]$ is prime. Let λ be an irreducible character of G. Then either the restriction of λ to H is irreducible or $\lambda = \chi^G$ for some irreducible character χ of H.

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[2] For a slightly different proof, one could appeal at this point to [2, (9.11)].

5. References

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