JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 78B, No. 1, January–March 1974

A Class of Positive Stable Matrices*

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(September 11, 1973)

A square complex matrix is positive sign-symmetric if all its principal minors are positive, and all products of symmetrically-placed minors are nonnegative. It is proved that every positive sign-symmetric matrix is positive stable.

Key words: Positive stable matrix; sign-symmetry; spectrum.

1. NOTATION. For fixed n, let

$$Q_n = \{(i_1, i_2, \ldots, i_k) \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}.$$

If $\alpha = (i_1, i_2, \ldots, i_k) \in Q_n$, then $|\alpha| = k$. Given $A \in C^{n,n}$ and $\alpha, \beta \in Q_n$, by $A(\alpha, \beta)$ we mean the minor of A whose rows are indexed by α and whose columns are indexed by β. We can now formally define $A \in C^{n,n}$ to be *positive sign-symmetric* if

(1) $A(\alpha, \alpha) > 0$ for all $\alpha \in Q_n$,
(2) $A(\alpha, \beta) A(\beta, \alpha) \geq 0$ for all $\alpha, \beta \in Q_n$, $|\alpha| = |\beta|$.

It is obvious that hermitian positive definite matrices and totally positive matrices are positive sign-symmetric. Also, it is well known that they have all positive characteristic roots (see also [9][1]). This last is not true for all positive sign-symmetric matrices; as an example, take

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 4 & 2 \\ 2 & 1 & 4 \end{pmatrix}.$$

The characteristic roots of this matrix are, approximately, 6.85 and $2.58 \pm 0.28i$.

It is perhaps of interest, however, that all positive sign-symmetric matrices are *positive stable*, i.e., all their characteristic roots have positive real parts.

THEOREM. *Every complex matrix which is positive sign-symmetric is positive stable.*

PROOF. Given $B \in C^{n,n}$ which is positive sign-symmetric. Let $S = B^2$; by the Cauchy-Binet formula and (1) and (2),

$$S(\alpha, \alpha) = \Sigma_{|\beta| = |\alpha|} B(\alpha, \beta) B(\beta, \alpha) > 0 \text{ for all } \alpha \in Q_n.$$

By a result of Fiedler and Pták [5], since all principal minors of S are positive, no real characteristic root of S is nonpositive. Since the characteristic roots of S are the squares of the characteristic roots of B, B can have no characteristic roots on the imaginary axis.

AMS Subject Classification: 15A18, 15A57, 65F15.
*An invited paper.
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[1] Figures in brackets indicate the literature references at the end of this paper.

Now let $A \epsilon C^{n,n}$ be a positive sign-symmetric matrix. Since all its principal minors are positive, by a result of Fisher and Fuller [6] and Ballantine [1], there exists a $D = \mathrm{diag}\,(d_1, d_2, \ldots, d_n) \; \epsilon C^{n,n}$, $d_i > 0$, $i = 1, \ldots, n$, for which AD is positive stable. Let $D_t = (1-t)\,I + tD$, $0 \leqslant t \leqslant 1$; all D_t are diagonal matrices, with positive diagonal entries, and $D_0 = I$, $D_1 = D$. It follows that each matrix AD_t is positive sign-symmetric, and hence can have no characteristic roots on the imaginary axis. Since AD is positive stable, and the characteristic roots of AD_t are continuous functions of t, A must also be positive stable.

We define $A \epsilon C^{n,n}$ to be *positive weakly sign-symmetric* if (1) and

(3) $A(\alpha,\beta)\,A\,(\beta,\alpha) \geqslant 0$ for all $\alpha, \beta \epsilon Q_n$, $|\alpha| = |\beta| = |\alpha \cap \beta| + 1$,

i.e., whenever all but one of the indices in α and β are common to both. A number of interesting inequalities are known to hold for the principal minors of such matrices; cf. [3], [4], [8]. It is known that nonsingular M-matrices, as well as hermitian positive definite and totally positive matrices, are positive weakly sign-symmetric (cf. [2]); and all these matrices are positive stable. We conjecture that all positive weakly sign-symmetric matrices are also positive stable. The Routh-Hurwitz conditions (cf. [7], p. 194–5) can be used to prove that the conjecture is true for $n \leqslant 4$ (the case $n = 3$ is also proved in [2]). An analogous proof for the general case appears hopelessly complicated.

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(Paper 78B1–391)