Second, Third, and Fourth Order D-Stability*

Charles R. Johnson

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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For \( n = 2, 3, \) and 4, conditions are given for the real \( n \) by \( n \) \( D \)-stable matrices. The \( 3 \) by \( 3 \) sufficient condition is easily checkable and reveals to be \( D \)-stable a class of matrices which is not included in any known, general sufficient condition.

Key words: \( D \)-stable; positive stable matrix; spectrum.

The concept of \( D \)-stability was originally introduced in the economic literature by Arrow and McManus \([1]\) with a stronger definition. We shall adopt the definition of fairly common current usage. Let \( M_n(R) \) denote the class of \( n \) by \( n \) matrices over the real field and denote by \( \sigma(A) \) the spectrum of \( A \in M_n(R) \). The matrix \( A \in M_n(R) \) is called \((positive)\) stable if \( \lambda \in \sigma(A) \) implies \( \Re(\lambda) > 0 \). We shall denote the multiplicative group of diagonal matrices with positive diagonal entries in \( M_n(R) \) by \( D_n \).

**Definition:** \( A \in M_n(R) \) is called \( D \)-stable if \( DA \) is stable for all \( D \in D_n \).

Several sufficient and some necessary conditions for \( D \)-stability are known; however, no general characterization is yet known. In this note we present conditions on the \( D \)-stable matrices in \( M_n(R) \) when \( n = 2, 3, \) and 4. Only one of the known necessary conditions will be of interest to us here.

**Definition:** \( A \in M_n(R) \) belongs to the class \( P_0 \) \([2]\) if and only if for each \( k = 1, \ldots, n \) all \( k \) by \( k \) principal minors of \( A \) are nonnegative. If also, at least one principal minor of each order is positive, then \( A \in P_0^+ \).

The best necessary condition for \( D \)-stability seems to be

**Theorem 0:** \([4, 5]\) If \( A \in M_n(R) \) is \( D \)-stable, then \( A \in P_0^+ \).

The converse of theorem 0 is, in general, far from valid. However, for \( n = 2 \) we have

**Theorem 1:** \( A \in M_2(R) \) is \( D \)-stable if and only if \( A \in P_0^+ \).

**Proof:** The necessity follows from theorem 0. Suppose \( A \in P_0^+ \cap M_2(R) \) and that \( D \) is an arbitrary element of \( D_2 \). Then \( DA \) has positive trace and positive determinant. Since \( DA \in M_2(R) \), this means that \( DA \) is positive stable and that \( A \) is \( D \)-stable which completes the proof.

For our remaining work we shall employ the stability theorem of Routh and Hurwitz \([3]\). For \( A \in M_n(R) \) denote the sum of the \( \binom{n}{k} \) principal minors of order \( k \) by \( E_k(A) \). Define the Routh-Hurwitz matrix \( \Omega \) by

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* This work was done while the author was a National Academy of Sciences—National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

* Figures in brackets indicate the literature references at the end of this paper.
Theorem 2: [Routh-Hurwitz] \( A \in M_n(\mathbb{R}) \) is positive stable if and only if the leading principal minors of \( \Omega(A) \) are positive.

Remark: \( A \in M_n(\mathbb{R}) \) is D-stable if and only if the leading principal minors of \( \Omega(DA) \) are positive for all \( D \in D_n \).

We are now in a position to give a numerical sufficient condition for 3 by 3 D-stability.

Theorem 3: The matrix \( A = \begin{bmatrix} X & \alpha & \beta \\ y & \gamma & z \\ a & b & c \end{bmatrix} \) is D-stable if (i) \( A \in P^+_3 \) and (ii) \( xyz > \frac{ac\beta + ayb}{2} \).

Proof: Since the conditions (i) and (ii) are preserved under multiplication from \( D_3 \), it suffices to show that they imply stability for which we shall use theorem 2. Conditions (i) and (ii) imply the positivity of the expression:

\[
(2xyz - ac\beta - ayb) + (x+y)(xy - aa) + (x+z)(xz - b\beta) + (y+z)(yz - cy)
\]

This is equivalent to the inequality

\[
E_1(A)E_2(A) > E_3(A).
\]

Because of (i) we also have that

\[
E_1(A) > 0.
\]

Together these mean that the leading principal minors of the 3 by 3 matrix \( \Omega(A) \) are positive which completes the proof.

The conditions of theorem 3 are easily checked for a given matrix. Theoretically they are of interest in that they reveal to be D-stable a class of 3 by 3 matrices which are not known to be D-stable by any other present sufficient condition [4].

Example: That the conditions of theorem 3 are not necessary for D-stability is shown, for instance, by letting \( A = \begin{bmatrix} 6 & 5 & -1 \\ 1 & 2 & 5 \\ 5 & -3 & 1 \end{bmatrix} \).
Then $A$ is $D$-stable since $A + A^*$ is positive definite [4]. However the inequality (ii) of theorem 3 is not satisfied since $12 > 64$.

We end with a characterization of $4$ by $4 D$-stability which, unfortunately, is not numerically checkable.

**Theorem 5**: $A \in M_4(\mathbb{R})$ is $D$-stable if and only if (i) $AeP_0^+$ and (ii) for each $D \in D_4$ such that $\det (DA) = 1$ we have

$$E_2(DA) > \frac{E_1(DA)}{E_3(DA)} + \frac{E_3(DA)}{E_1(DA)}.$$  

**Proof**: By theorem 0 we know that the $D$-stability of $A$ implies $AeP_0^+$. We thus assume $AeP_0^+$ and show that $A$ is $D$-stable if and only if condition (ii). However $A \in M_4(\mathbb{R})$ is $D$-stable if and only if $\Omega(DA)$ has positive leading principal minors for $D \in D_4$. Under the assumption $E_3(DA) = 1$ which provides no loss of generality this is equivalent to $E_1(DA) > 0$, $E_2(DA)E_1(DA) > E_3(DA)$, and $E_1(DA)E_2(DA)E_3(DA) > E_1(DA)^2 + E_3(DA)^2$. The first of these conditions is subsumed in the assumption $AeP_0^+$ and the second is subsumed in the third which is equivalent to (ii). This completes the proof.

In considering sufficient conditions for or characterizations of $D$-stability one of course wishes conditions which are invariant under multiplication from $D_n$. This is a virtue of the new condition (ii) of theorem 3. Whether or not there are significant generalizations of theorem 3 is worthy of further study.

**References**


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