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Second, Third, and Fourth Order *D*-Stability*

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For $n=2$, 3, and 4, conditions are given for the real n by n D-stable matrices. The 3 by 3 sufficient condition is easily checkable and reveals to be D-stable a class of matrices which is not included in any known, general sufficient condition.

Key words: D-stable; positive stable matrix; spectrum.

The concept of D-stability was originally introduced in the economic literature by Arrow and McManus [1][1] with a stronger definition. We shall adopt the definition of fairly common current usage. Let $M_n(R)$ denote the class of n by n matrices over the real field and denote by $\sigma(A)$ the *spectrum* of $A\epsilon M_n(R)$. The matrix $A\epsilon M_n(R)$ is called (*positive*) *stable* if $\lambda\epsilon\sigma(A)$ implies $Re(\lambda)>0$. We shall denote the multiplicative group of diagonal matrices with positive diagonal entries in $M_n(R)$ by D_n.

DEFINITION: $A\epsilon M_n(R)$ *is called* D-*stable if* DA *is stable for all* $D\epsilon D_n$.

Several sufficient and some necessary conditions for D-stability are known; however, no general characterization is yet known. In this note we present conditions on the D-stable matrices in $M_n(R)$ when $n=2$, 3, and 4. Only one of the known necessary conditions will be of interest to us here.

DEFINITION: $A\epsilon M_n(R)$ *belongs to the class* P_0 [2] *if and only if for each* $k=1, \ldots, n$ *all* k *by* k *principal minors of* A *are nonnegative. If also, at least one principal minor of each order is positive, then* $A\epsilon P_0^+$.

The best necessary condition for D-stability seems to be

THEOREM 0: [4, 5] *If* $A\epsilon M_n(R)$ *is* D-*stable, then* $A\epsilon P_0^+$.

The converse of theorem 0 is, in general, far from valid. However, for $n=2$ we have

THEOREM 1: $A\epsilon M_2(R)$ *is* D-*stable if and only if* $A\epsilon P_0^+$.

PROOF: The necessity follows from theorem 0. Suppose $A\epsilon P_0^+\cap M_2(R)$ and that D is an arbitrary element of D_2. Then DA has positive trace and positive determinant. Since $DA\epsilon M_2(R)$, this means that DA is positive stable and that A is D-stable which completes the proof.

For our remaining work we shall employ the stability theorem of Routh and Hurwitz [3]. For $A\epsilon M_n(R)$ denote the sum of the $\binom{n}{k}$ principal minors of order k by $E_k(A)$. Define the Routh-Hurwitz matrix Ω by

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[1] Figures in brackets indicate the literature references at the end of this paper.

$$
\Omega(A)=\begin{bmatrix}
E_1(A) & E_3(A) & E_5(A) & \cdots & 0\\
1 & E_2(A) & E_4(A) & \cdots & 0\\
0 & E_1(A) & E_3(A) & \cdots & 0\\
0 & 1 & E_2(A) & \cdots & 0\\
\vdots & \vdots & \vdots & & \vdots\\
0 & 0 & 0 & \cdots & E_n(A)
\end{bmatrix}.
$$

THEOREM 2: [Routh-Hurwitz] $A \in M_n(R)$ *is positive stable if and only if the leading principal minors of* $\Omega(A)$ *are positive.*

REMARK: $A \in M_n(R)$ is D-stable if and only if the leading principal minors of $\Omega(DA)$ are positive for all $D \in D_n$.

We are now in a position to give a numerical sufficient condition for 3 by 3 D-stability.

THEOREM 3: *The matrix* $A=$

$$
\begin{bmatrix}
x & a & b\\
\alpha & y & c\\
\beta & \gamma & z
\end{bmatrix} \in M_3(R)
$$

is D-*stable if* (*i*) $A \in P_0^+$ *and* (*ii*) $xyz > \dfrac{ac\beta + \alpha\gamma b}{2}$.

PROOF: Since the conditions (i) and (ii) are preserved under multiplication from D_3, it suffices to show that they imply stability for which we shall use theorem 2. Conditions (i) and (ii) imply the positivity of the expression:

$$(2xyz - ac\beta - \alpha\gamma b) + (x+y)\,(xy - a\alpha) + (x+z)\,(xz - b\beta) + (y+z)\,(yz - c\gamma)$$

This is equivalent to the inequality

$$E_1(A)E_2(A) > E_3(A).$$

Because of (i) we also have that

$$E_1(A) > 0$$

Together these mean that the leading principal minors of the 3 by 3 matrix $\Omega(A)$ are positive which completes the proof.

The conditions of theorem 3 are easily checked for a given matrix. Theoretically they are of interest in that they reveal to be D-stable a class of 3 by 3 matrices which are not known to be D-stable by any other present sufficient condition [4].

EXAMPLE: That the conditions of theorem 3 are not necessary for D-stability is shown, for instance, by letting $A=$

$$
\begin{bmatrix}
6 & 5 & -1\\
1 & 2 & 5\\
5 & -3 & 1
\end{bmatrix}.
$$

Then A is D-stable since $A+A^*$ is positive definite [4]. However the inequality (ii) of theorem 3 is not satisfied since $12 \not> 64$.

We end with a characterization of 4 by 4 D-stability which, unfortunately, is not numerically checkable.

THEOREM 5: $A \epsilon M_4(R)$ *is* D-*stable if and only if* (*i*) $A \epsilon P_0^+$ *and* (*ii*) *for each* $D \epsilon D_4$ *such that det* $(DA)=1$ *we have*

$$E_2(DA) > \frac{E_1(DA)}{E_3(DA)} + \frac{E_3(DA)}{E_1(DA)}.$$

PROOF: By theorem 0 we know that the D-stability of A implies $A \epsilon P_0^+$. We thus assume $A \epsilon P_0^+$ and show that A is D-stable if and only if condition (ii). However $A \epsilon M_4(R)$ is D-stable if and only if $\Omega(DA)$ has positive leading principal minors for $D \epsilon D_4$. Under the assumption $E_4(DA)=1$ which provides no loss of generality this is equivalent to $E_1(DA)>0$, $E_2(DA)E_1(DA)>E_3(DA)$, and $E_1(DA)E_2(DA)E_3(DA)>E_1(DA)^2+E_3(DA)^2$. The first of these conditions is subsumed in the assumption $A \epsilon P_0^+$ and the second is subsumed in the third which is equivalent to (ii). This completes the proof.

In considering sufficient conditions for or characterizations of D-stability one of course wishes conditions which are invariant under multiplication from D_n. This is a virtue of the new condition (ii) of theorem 3. Whether or not there are significant generalizations of theorem 3 is worthy of further study.

References

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