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Character Induced Subgroups*

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Given a finite group G with an irreducible character χ, define $G_\chi = \{g \epsilon G : |\chi(g)| = \chi(1)\}$. Then $\chi(1)^2 \leq [G: G_\chi]$. We investigate the case of equality. There are applications to symmetry classes of tensors and generalized matrix functions.

Key words: Character of central type; central idempotent; Frobenius Reciprocity; group algebra; orthogonality relations; tensor product.

1. Character Induced Subgroups

Let G be a finite group with an irreducible (complex) representation $\{A(g) : g \epsilon G\}$ and corresponding character χ. Let

$$G_\chi = \{g \epsilon G : |\chi(g)| = \chi(1)\}.$$

One easily sees that G_χ is the normal subgroup of G consisting of those elements represented by scalars in $\{A(g)\}$. Moreover, $\lambda = \chi/\chi(1)$ is a linear character on G_χ which is invariant under conjugation by elements of G. we call G_χ the subgroup *induced* by χ.

THEOREM 1: *We have* $\chi(1)^2 \leq [\mathrm{G} : \mathrm{G}_\chi]$, *with equality if and only if* χ *is the only irreducible character of* G *whose restriction to* G_χ *contains* λ *as a component.*

PROOF: Let λ^G be the character of G induced by λ. Then, by the Frobenius Reciprocity Theorem, χ occurs in λ^G exactly $\chi(1)$ times. Moreover, if η is an irreducible character on G whose restriction to G_χ contains λ, then $\eta \epsilon \lambda^G$. But, the degree of λ^G is $[G : G_\chi]$.

We might point out that if η is an irreducible character on G such that $\lambda \epsilon \eta \mid G_\chi$, then $\eta \mid G_\chi = \eta(1)\lambda$ [8, p. 53]. In particular, $G_\chi \subset G_\eta$.

COROLLARY 1: *Let* $\mathrm{g} \epsilon \mathrm{G}$ *be arbitrary. Then* λ *can be extended to a character of* $\langle \mathrm{G}_\chi, \mathrm{g}\rangle$, *the group generated by* G_χ *and* g. *If* $\chi(1)^2 = [\mathrm{G} : \mathrm{G}_\chi > 1$, *then* λ *cannot be extended to* G.

PROOF: The first statement follows because G_χ is normal and λ is invariant. The second follows from Theorem 1.

We now give another proof of Theorem 1 which leads to an apparently different case of equality. First, define the *support of* χ to be $\mathrm{supp}\,\chi = \{g \epsilon G : \chi(g) \neq 0\}$.

THEOREM 1′: *We have* $\chi(1)^2 \leq [\mathrm{G} : \mathrm{G}_\chi]$, *with equality if and only if supp* $\chi = \mathrm{G}_\chi$.

PROOF: Let $G = \bigcup_{i=1}^{m} g_i G_\chi$ be the coset decomposition of G with respect to G_χ. Then

$$o(G) = \sum_{g \epsilon G} |\chi(g)|^2$$

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$$= \sum_{i=1}^{m} \sum_{h \epsilon G_\chi} | \chi(g_i) |^2$$

$$= o(G_\chi) \sum_{i=1}^{m} | \chi(g_i) |^2.$$

The case of equality in Theorem 1′ recalls the following result of Brunside

THEOREM 2: ([3, p. 322], [8, p. 93], [11, p. 41]). *Suppose that* $(o(C), \chi(1)) = 1$ *for some conjugacy class* C *of* G. *Then* $\chi(C)=0$ or $|\chi(C)|=\chi(1)$.

THEOREM 3: *If* $A_1, \ldots, A_r$ *are the inequivalent irreducible representations of* G *and if* G_i *is the subgroup induced by the character of* A_i, *then*

$$\bigcap_{i=1}^{r} G_i = Z(G), \tag{1}$$

the center of G. *If* A_i *is faithful, then* $G_i = Z(G)$.

PROOF: The group G_i is the inverse image of the center of $\{A_i(g)\}$. Equation (1) follows from an examination of the center of the regular representation.

THEOREM 4: *Suppose* $\chi(1)^2 = [G : G_\chi]$. *Let* $K = \{g \epsilon G : \chi(g) = \chi(1)\}$. *Then* $(G/K)_\chi \cong G_\chi/K$ *is cyclic.*

PROOF: If $G = \bigcup_{i=1}^{m} g_i K$ is a coset decomposition of G, $g_i K \to A(g_i)$ defines an irreducible representation of G/K. Thus, χ is a character of G/K and $(G/K)_\chi \cong G_x/K$. But, G_χ/K is isomorphic to the group of values of λ, a subgroup of the $o(G_\chi)$th roots of unity.

The group G is of *central type* if $\chi(1)^2 = [G:Z(G)]$ for some irreducible character χ [6], [7]. Since $Z(G) \subset G_\chi$, if G is of central type with respect to χ, then $\chi(1)^2 = [G : G_\chi]$.

THEOREM 5: *If* $\chi(1)^2 = [G : G_\chi]$, *then there exist irreducible characters* $\chi_p, \chi_q, \ldots, \chi_r$ *on the Sylow subgroups* $S_p, S_q, \ldots, S_r$ *of* G, *respectively, such that* $(S_t)_{\chi_t} = S_t \cap G_\chi$, $t = p, q, \ldots, r$, *and* $[S_t : (S_t)_{\chi_t}] = \chi_t(1)^2$, $t = p, q, \ldots, r$.

The following simple proof of Theorem 5 was communicated by Professor DeMeyer to the first author [5]: If $\chi(1)^2 = [G : G_\chi]$, then $G/\ker \chi$ is a group of central type with character χ. Thus, Theorem 5 follows from [7, Theorem 2].

2. Examples

(a) If χ is linear, then $G_\chi = G$ and $\chi(1)^2 = [G : G_\chi]$.

(b) Let G be a finite group with normal subgroup H. Let $\{B(gH) : g \epsilon G\}$ be any irreducible representation of G/H. Then $\{B(gH)\}$ gives rise to an irreducible representation of G, $A(g) = B(gH)$. Let χ be the character afforded by $\{A(g)\}$. Since H is in the kernel of $\{A(g)\}$, $H \subset G_\chi$.

(c) Let G be a finite group with irreducible character χ. Let L be any group with linear character η. Then $H = G \times L$ is endowed with the irreducible character ζ, $\zeta(g, 1) = \chi(g)\,\eta(1)$. Moreover, $H_\zeta = G_\chi \times L$, and $[H : H_\zeta] = [G \times L : G_\chi \times L] = [G : G_\chi]$.
If $[G : G_\chi] = \chi(1)^2$ then $[H : H_\zeta] = \zeta(1)^2$. If L is not abelian, then $Z(H) \subsetneq H_\zeta$.

(d) Let p be a prime. Suppose G is the group of order p^3 generated by elements g and h with defining relations

$$g^{p^2} = h^p = 1, h^{-1}gh = g^{p+1}.$$

Let A be the representation of G of degree p given by

$$A(g) = \omega^{1/p} \operatorname{diag}(1, \omega, \ldots, \omega^{p-1}),$$

$$A(h) = \begin{bmatrix} 0 & 0 & . & . & . & 0 & 1 \\ 1 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & . & . & . & 0 & 0 \\ & & & . & . & . & \\ 0 & 0 & . & . & . & 1 & 0 \end{bmatrix}$$

where $\omega = \exp(2\pi i / p)$. Let χ be the character afforded by A. Then, $G_\chi = Z(G) = \langle g^p \rangle$ is of order p, and $\chi(g^p) = p\omega$. (For details, see [11, P. 43].)

3. Central Idempotents

The authors believe the material of this section to be of independent interest, but it may also be viewed as lemmata for Theorem 8 in the next section.

Let KG denote the complex group algebra of G. Define

$$t(G, \chi) = \frac{\chi(1)}{o(G)} \sum_{g \epsilon G} \chi(g)g.$$

Suppose $\chi = \chi_1, \ldots, \chi_r$ are inequivalent, irreducible characters on G. Then $t(G, \chi_1), \ldots, t(G, \chi_r)$ are annihilating idempotents and

$$t(G, \chi_1) + \ldots + t(G, \chi_r) = 1 \epsilon G.$$

Moreover, $t(G, \chi_1), \ldots, t(G, \chi_r)$ span $Z(KG)$, the center of KG. Finally, $t(G, \chi_i)$ generates the simple two sided ideal in KG to which χ_i corresponds [4, pp. 233–236], [1, p. 83].

THEOREM 6: *We have* $t(G_\chi, \lambda) = \Sigma t(G, \eta)$, *where the summation is over those inequivalent, irreducible characters* η *of* G *whose restriction to* G_χ *contains* λ *as a component. In particular,* $t(G_\chi, \lambda) = t(G, \chi)$ *if and only if* $[G : G_\chi] = \chi(1)^2$.

PROOF: We claim first, that $t(G_\psi, \lambda) \epsilon Z(KG)$. Take $g \epsilon G$. Then

$$\begin{aligned} g^{-1} t(G_\chi, \lambda) g &= \frac{1}{o(G_\chi)} \sum_{h \epsilon G_\chi} \lambda(h) g^{-1} h g \\ &= \frac{1}{o(G_\chi)} \sum_{h \epsilon G_\chi} \lambda(g h g^{-1}) h \\ &= t(G_\chi, \lambda), \end{aligned}$$

because G_χ is normal in G and λ is invariant under conjugation by elements of G.

Thus, there exist complex numbers $a_1, \ldots, a_r$ such that

$$t(G_\chi, \lambda) = \sum_{i=1}^{r} a_i t(G, \chi_i). \qquad (2)$$

Considered as linear operators on KG (multiply on the left), $t(G, \chi_i)$, and $t(G_\chi, \lambda)$ are hermitian with respect to the inner product which makes $G \subset KG$ an $o.\ n.$ basis. Since they are also idempotent, we may view them as orthogonal projections on KG. Since $t(G, \chi_1), \ldots, t(G, \chi_r)$ are mutually annihilating and belong to $Z(KG)$, all the a_i's must be 1 or 0. Indeed,

$$a_i = \begin{cases} 1, \text{ if } t(G_\chi, \lambda)t(G, \chi_i) = t(G, \chi_i), \\ 0 \text{ otherwise.} \end{cases}$$

Thus, $a_i = 1$ if and only if $t(G_\chi, \lambda)t(G, \chi_i)$ is not zero. But, if ζ is the character of the regular representation of G,

$$\begin{aligned} &\text{rank } (t(G_\chi, \lambda)t(G, \chi_i)) \\ &= \text{trace } (t(G_\chi, \lambda)t(G, \chi_i)) \\ &= \frac{\chi_i(1)}{o(G_\chi)o(G)} \sum_{h \in G_\chi} \lambda(h) \sum_{g \in G} \chi_i(h^{-1}g)\zeta(g) \\ &= \frac{\chi_i(1)}{o(G_\chi)} \sum_{h \in G_\chi} \lambda(h)\chi_i(h^{-1}). \end{aligned}$$

This last expression is $\chi_i(1)$ times the number of occurrences of λ in $\chi_i|G_\chi$.

4. Symmetry Classes of Tensors

Let V be an n-dimensional complex inner product space. Take $m \leq n$. Denote by $\otimes^m V$ the mth tensor power of V. If $v_1, \ldots, v_m \in V$, then $v_1 \otimes \ldots \otimes v_m \in \otimes^m V$ will denote their tensor product. If $(.,.)$ denotes the inner product on V, then

$$(v_1 \otimes \ldots \otimes v_m, v'_1 \otimes \ldots \otimes v'_m) = \prod_{t=1}^{m} (v_t, v'_t) \tag{3}$$

is an inner product on $\otimes^m V$.

Let S_m denote the symmetric group of degree m. Given $g \in S_m$, there is a linear operator, $P(g)$ on $\otimes^m V$ such that

$$P(g^{-1})v_1 \otimes \ldots \otimes v_m = v_{g(1)} \otimes \ldots \otimes v_{g(m)},$$

for all $v_1, \ldots, v_m \in V$. Observe that $g \to P(g)$ is a representation of S_m.

Suppose, now, that G is a subgroup of S_m and χ is an irreducible character on G. Let

$$T(G, \chi) = \frac{\chi(1)}{o(G)} \sum_{g \in G} \chi(g)P(g).$$

With respect to the inner product, (3), $T(G, \chi)$ is hermitian. It is a consequence of the orthogonality relations for characters that $T(G, \chi)$ is idempotent. Thus, $T(G, \chi)$ is an orthogonal projection. Let $V_\chi^m(G)$ denote the range of $T(G, \chi)$. Then $V_\chi^m(G)$ is called a *symmetry class of tensors.*

Let V^m denote the mth cartesian power of V. Define

$$f: V^m \to V_\chi^m(G) \text{ by } f(v_1, \ldots, v_m) = T(G, \chi)v_1 \otimes \ldots \otimes v_m.$$

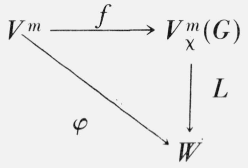
clearly, f is m-linear. If χ is a linear character, then f is symmetric with respect to G and χ, i.e., if $g \in G$, then

$$f(v_{g(1)}, \ldots, v_{g(m)}) = \chi(g) f(v_1, \ldots, v_m). \tag{4}$$

This follows from the observation that

$$T(G, \chi) P(g^{-1}) = \chi(g) T(G, \chi). \tag{5}$$

From (4), and the Universal Factorization Property for tensor spaces, it follows that: If W is any complex vector space and φ: $V^m \to W$ is any m-linear function, symmetric with respect to G and χ, then there exists a unique linear L: $V_\chi^m(G) \to W$ such that $\varphi = L \cdot F$, i.e., such that the diagram



is commutative.

For characters of degree greater than one, (5) does not hold. We are motivated to define

$$G' = \{g \in S_m \colon T(G, \chi) P(g^{-1}) \in \langle T(G, \chi) \rangle\},$$

i.e., G' is the set of elements $g \in S_m$ such that $T(G, \chi) P(g^{-1})$ is a multiple of $T(G, \chi)$. For $g \in G'$, write

$$T(G, \chi) P(g^{-1}) = c(g) T(G, \chi).$$

THEOREM 7: *We have* $G' = G_\chi$ *and* $c = \lambda$.

PROOF: Suppose, first, that $g \in G_\chi$. Then

$$\begin{aligned} T(G, \chi) P(g^{-1}) &= \frac{\chi(1)}{o(G)} \sum_{h \in G} \chi(h) P(h) P(g^{-1}) \\ &= \frac{\chi(1)}{o(G)} \sum_{h \in G} \chi(hg) P(h) \\ &= \lambda(g) T(G, \chi). \end{aligned}$$

Thus, $G_\chi \subset G'$ and $c | G_\chi = \lambda$.

Next, we prove that G' is a group and c is a linear character on it.

Since $P(1)$ is the identity, $1 \in G'$ and $c(1) = 1$. Since $P(g^{-1}) = P(g)^{-1}$, $g \in G'$ implies $g^{-1} \in G'$ and $c(g^{-1}) = c(g)^{-1}$. Finally, if $g, h \in G'$, then

$$\begin{aligned} T(G, \chi) P(h^{-1} g^{-1}) &= c(h) T(G, \chi) P(g^{-1}) \\ &= c(h) c(g) T(G, \chi). \end{aligned}$$

It follows that $gh \in G'$ and $c(gh) = c(g) c(h)$.

Now let $g \in G'$. Let $v_1, \ldots, v_m$ be an *o.n.* set in V. We compute the same inner product in two ways to obtain

$$(f(v_{g(1)}, \ldots, v_{g(m)}), c(g) f(v_1, \ldots, v_m))$$

$$= \overline{c(g)} \frac{\chi(1)}{o(G)} \sum_{h \in G} \chi(h) \prod_{t=1}^{m} (v_{g(t)}, v_{h(t)})$$

$$= \begin{cases} \overline{c(g)} \dfrac{\chi(1)}{o(G)} \chi(g), \ g \in G \\ 0, \ g \notin G, \end{cases}$$

and $||c(g) f(v_1, \ldots, v_m)||^2 = \chi(1)^2/o(G)$. Thus,

$$G' \subset \{g \in G \colon |\chi(g)| = \chi(1)\} = G_\chi.$$

Our next result involves the generalized matrix functions of Schur [12]. If $A = (a_{ij})$ is an m-square complex matrix, define

$$d_G^\chi(A) = \sum_{g \in G} \chi(g) \prod_{t=1}^{m} a_{tg(t)}.$$

THEOREM 8: *Let* $A = (a_{ij})$ *be an* m*-square matrix. Then*

$$\frac{1}{o(G_\chi)} d_{G_\chi}^\lambda(A) = \sum \frac{\eta(1)}{o(G)} d_G^\eta(A),$$

where the summation is over those inequivalent, irreducible characters η *of* G *whose restriction to* G_χ *contains* λ *as a component.*

When $G_\chi = \{1\}$, this result collapses to an identity of Freese [9, eq (8)].

COROLLARY 2: *Suppose* A *is positive semidefinite hermitian.*
Then

$$\frac{1}{o(G_\chi)} d_{G_\chi}^\lambda(A) \geqslant \frac{\chi(1)}{o(G)} d_G^\chi(A).$$

If A *is positive definite, equality obtains if and only if* $[G \colon G_\chi] = \chi(1)^2$.

PROOF: Schur proved [11] that $d_G^\eta(A) \geqslant 0$, with strict inequality if A is positive definite. Thus, the result is an immediate consequence of the theorem.

The inequality in Corollary 2 is of a type recently studied by Botta [2] and others.

PROOF of THEOREM 8: Extend $\sigma \to P(\sigma)$ linearly to a representation $\bar{P}$ of KG. Then $\bar{P}$ is a homomorphism of the algebra KG onto the operator algebra generated by $\{P(g) \colon g \in G\}$.

(Indeed, since $m \leqslant n$, $V_\eta^m(G) \neq 0$ for every irreducible character η of G [10, Lemma 1]. But, the dimension of $V_\eta^m(G)$ is trace $T(G, \eta)$, which is the number of occurrences of η in the character of the representation $g \to P(g)$. Thus, every irreducible character of G is a component of the character $\{\operatorname{tr} P(g)\}$. It follows that $\bar{P}$ is, in fact, an isomorphism [1, p. 69].)

Since $T(G, \chi_i) = \bar{P}(t(G, \chi_i))$, we may apply Theorem 6 to obtain

$$T(G_\chi, \lambda) = \sum T(G, \eta) \tag{6}$$

where the summation is as it was there.

Let, now, $v_1, \ldots, v_m, v_1', \ldots, v_m'$ be vectors such that $a_{ij} = (v_i, v_j')$. Observe that

$$(T(G, \eta) v_1 \otimes \ldots \otimes v_m, v_1' \otimes \ldots \otimes v_m') =$$
$$= \frac{\eta(1)}{o(G)} \sum_{g \in G} \eta(g) \prod_{t=1}^{m} (v_t, v_{g(t)}')$$

$$=\frac{\eta(1)}{o(G)} d_G^\eta(A).$$

The result follows from (6).

We might remark that eq (6) also shows that

$$V_\lambda^m(G_\chi) = \perp V_\eta^m(G),$$

the orthogonal direct sum of those symmetry classes corresponding to the irreducible characters η of G whose restriction to G_χ contains λ.

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