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The Characterizations of $(A_q(U))$*

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For $q>1$ Bers defines a Banach space $A_q(U)=\left\{f\epsilon H(U)\,|\int_U |f(u)|(1-|u|^2)^{q-2}dxdy<\infty\right\}$ and shows that any bounded linear functional Λ on $A_q(U)$ may be represented as $\Lambda(f)=\int_U f(u)\bar{G}(u)(1-|u|^2)^{2q-2}dxdy$ where $G\epsilon B_q(U)=\{h\epsilon H(U)|\sup_{u\epsilon U}|h(u)|(1-|u|^2)^q<\infty\}$ and is unique. This work is done in [1][1]. Duren, Romberg, and Shields, pursuant to their work on H^p for $p<1$, define a Banach space $B^p(p<1)=\left\{f\epsilon H(U)\,|\int_0^{2\pi}\int_0^1 |f(re^{i\theta})|(1-r)^{1/p-2}drd\theta<\infty\right\}$. They show that a bounded linear functional Λ on B^p may be uniquely represented as $\Lambda(f)=\lim_{r\to 1}\int_0^{2\pi} f_r(e^{i\theta})\bar{g}(e^{i\theta})d\theta$ where

(i) $f_r(e^{i\theta})=f_r(re^{i\theta})$
(ii) $g\epsilon A=\{h\epsilon H(U)|$ g is continuous on $\bar{U}\}$ and $g^{(n-1)}$, and $n-1$ *st* derivative of g, is in $\Lambda_\alpha=\{h\epsilon H(U)|h'(re^{i\theta})=0(1-r)^{\alpha-1})\}$. (Here $\alpha=1/p-n$ where $n<1/p<n+1$, so $\alpha\neq 0$. If $1/p=n+1$, the conditions on g are: $g\epsilon A$, $g^{(n-1)}\epsilon\Lambda_*=\{h\epsilon H(U)|h''(re^{i\theta})=0((1-r)^{-1})\}$.) This work appears in [2].

In this paper, after showing that $B^p\cong A_q(U)$ with $1/p=q$, we derive the relationship between G and g, namely:

$$G(z)=\sum_{k=0}^{2n+1} A_k\cdot g^{(2k+1)}(z)\cdot z^{2k+1},$$

$(|z|<1)$ where A_i are constants, $A_{2n+1}\neq 0$. (in this case $q=1/p$ is an integer. The Theorem is slightly different if q is not an integer.)

Key words: Automorphic functions; Hardy spaces.

We begin by showing that B^p is isomorphic to $A_q=A_q(U)$ with $1/p=q$. (the mapping is the identity.)

PROOF: Take $f\epsilon B^p$.

$$\int_U |f|(1-|u|^2)^{q-2}dxdy=\frac{1}{2\pi}\int_0^1 |f|(1-r^2)^{q-2}rdrd\theta\leqslant\frac{\max\,(1,2^{q-2})}{2\pi}\int_0^{2\pi}\int_0^1 |f|(1-r)^{q-2}drd\theta.$$

This shows that $k\|f\|_{B^p}\geqslant\|f\|_{A_q}$. So $B^p\subset A_q$, with a continuous injection. Now it is easy to verify that if $f\epsilon A_q$ then $f\epsilon B^p$. Thus we have $i:B^p\to A_q$, i the injection, is continuous and onto. The Open Mapping Theorem implies that i^{-1} is continuous, completing the proof that $B^p\cong A_q$, with $1/p=q$.

We turn to the Duren, Romberg and Shields representation. We hereafter will only refer to A_q, dropping all reference to B^p. If $f(z)=z^n$, $n\geqslant 0$ (which is certainly in A_q).

$$\Lambda(z^n)=\lim_{r\to 1}\int_0^{2\pi} r^n e^{in\theta}\bar{g}(e^{i\theta})d\theta=\int_0^{2\pi} e^{in\theta}\bar{g}(e^{i\theta})d\theta,$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

since $g \epsilon A$.

But now

$$\int_0^{2\pi} e^{in\theta}\bar{g}(e^{i\theta}) = \lim_{r\to 1}\int_0^{2\pi} e^{in\theta}\bar{g}_r(e^{i\theta})d\theta,$$

by the Lebesgue Dominated Convergence Theorem since $\|g_r\|_\infty \leq \|g\|_\infty$. Now, letting $g(z) = \sum_{k=0}^{\infty} b_k z^k$,

$$\int_0^{2\pi} e^{in\theta}\bar{g}_r(e^{in\theta})d\theta = \int_0^{2\pi} e^{in\theta}\sum_0^\infty \bar{b}_k r^k e^{-ik\theta}d\theta$$

$$= \sum_0^\infty \int_0^{2\pi} (\bar{b}_k r^k e^{in\theta - ik\theta})d\theta$$

(since the power series for g_r converges absolutely in $\bar{U}$)

$$= 2\pi \bar{b}_n r^n$$

and

$$\lim_{r\to 1}\int_0^{2\pi} e^{in\theta}\bar{g}_r(e^{i\theta})d\theta = 2\pi\bar{b}_n.$$

We have proved

LEMMA 1: *If* $g(z) = \sum_{k=0}^{\infty} b_k z^k$,

$$\Lambda(z^n) = 2\pi\bar{b}_n.$$

Now we turn to the Bers representation of $\Lambda(z^n)$. *If*

$$G(z) = \sum a_k z^k,\ \lambda(z) = (1-|z|^2)$$

$$\Lambda(z^n) = \int\int_U z^n \bar{G}(z)\lambda^{2q-2}dxdy$$

$$= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} r^n e^{in\theta}\sum_0^\infty \bar{a}_k r^k e^{-ik\theta}(1-r^2)^{2q-2} r dr d\theta.$$

Define $G_\rho(z) = G(\rho z)$. Then

$$\sup_{|z|=s} |G_\rho(z)\lambda^q(z)| = \sup_{|z|=s} |G_\rho(z)|\lambda^q(s)$$

(since λ depends only on $|z|$)

$$\leq \sup_{|z|=s} |G(z)|\lambda^q(s)$$

$$= \sup_{|z|=s} |G(z)|\lambda^q(z).$$

We have shown that

$$|G_\rho(z)\lambda^q(z)| \leqslant \sup_{|\zeta|\leqslant|z|} |G_\rho(\zeta)\lambda^q(\zeta)|$$

$$\leqslant \sup_{|\zeta|\leqslant|z|} |G(\zeta)\lambda^q(\zeta)| \leqslant K;$$

since $G \epsilon B_q$. Then,

(*) $$\lim_{\rho\to 1}\int\int f\bar{G}_\rho\lambda^{2q-2}dxdy=\int\int f\bar{G}\lambda^{2q-2}dxdy$$

since

$$|fG_\rho\lambda^{2q-2}|=|f(G_\rho\lambda^q)\lambda^{q-2}|\leqslant K|f\lambda^{q-2}|,$$

by above. But since $f\epsilon A_q$, $K|f\lambda^{q-2}|$ is integrable so (*) holds. Now, as before, we compute $\int\int z^n\bar{G}_\rho\lambda^{2q-2}dxdy, n\geqslant 0$.

$$\int\int z^n\bar{G}_\rho\lambda^{2q-2}dxdy$$

$$=\frac{1}{2\pi}\int_0^1\int_0^{2\pi} r^n e^{in\theta}\left(\sum_0^\infty \bar{a}_k\rho^k r^k e^{-ik\theta}\right)(1-r^2)^{2q-2}rdrd\theta$$

$$=\frac{1}{2\pi}\int_0^1 (1-r^2)^{2q-2}\cdot r^{n+1}\cdot\int_0^{2\pi} e^{in\theta}\left(\sum_0^\infty a_k\rho^k r^k e^{-ik\theta}\right)d\theta dr$$

$$=\frac{1}{2\pi}\int_0^1 (1-r^2)^{2q-2}r^{n+1}\sum_0^\infty\int_0^{2\pi} \bar{a}_k\rho^k r^k e^{in\theta-ik\theta}d\theta dr$$

(since the series for G_ρ converges absolutely for $|z|=r\leqslant 1$)

$$=\frac{1}{2\pi}\int_0^1 (1-r^2)^{2q-2}r^{n+1}\bar{a}_n\rho^n r^n\cdot 2\pi dr$$

$$=\frac{1}{2\pi}\cdot 2\pi\bar{a}_n\rho^n\int_0^1 (1-r^2)^{2q-2}r^{2n+1}dr$$

and taking limit as $\rho\to 1$ we get

Lemma 2: *If* $G(z)=\sum_0^\infty a_k z^k$

$$\Lambda(z^n)=\bar{a}_n\cdot\int_0^1 (1-r^2)^{2q-2}r^{2n+1}dr=\bar{a}_n\cdot c_n$$

where

$$c_n=\int_0^1 (1-r^2)^{2q-2}r^{2n+1}dr.$$

COROLLARY 3: $n \geqslant 0$

$$2\pi b_n = a_n c_n.$$

PROOF: By taking conjugates, and noting that c_n and 2π are real.

We shall need the following computation.

LEMMA 4: *Let*

$$_p c_n = \int_0^1 r^{2n+1}(1-r^2)^p dr,$$

then

$$_p c_n = \frac{n!\Gamma(p+1)}{\Gamma(p+n+2)\cdot 2}.$$

(Here $p \neq -1$.)

PROOF: By integration by parts

$$_p c_n = \int_0^1 r^{2n+1}(1-r^2)^p dr$$

$$= \frac{1}{p+1}\int r^{2n-1}(1-r^2)^{p+1} dr$$

$$= \frac{n}{p+1}\,{}_{(p+1)}c_{(n-1)}.$$

Also, it is easy to compute

$$_p c_0 = \frac{1}{2(p+1)}.$$

Using our recursion equation and initial condition, we obtain:

$$(**) \qquad _p c_n = \frac{n!}{2(p+1)\ .\ .\ .\ (p+n+1)}.$$

Now

$$p\Gamma(p) = \Gamma(p+1).$$

So

$$[2(p+1)\ .\ .\ .\ (p+n+1)]\Gamma(p+1) = 2\Gamma(p+n+2)$$

or

$$[2(p+1)\ .\ .\ .\ (p+n+1)] = \frac{2\Gamma(p+n+2)}{\Gamma(p+1)}.$$

Substituting in (**), we obtain

$$_p c_n = \frac{n!\Gamma(p+1)}{2\Gamma(p+n+2)}.$$

We can now prove the basic theorems. First, the case when $q = n+1$, an integer.

THEOREM 5: *Let* $q=n+1$, *then*

(1) $G(z)=A_{2n+1}g^{(2n+1)}(z)\cdot z^{2n+1}+A_{2n}g^{(2n)}(z)\cdot z^{2n}+\ldots+A_0 g(z)(|z|<1)$ *where* A_i *are constants*, $A_{2n+1}\neq 0$.

PROOF: $a_m=2\pi b_m/c_m$ where

$$c_m=(2q-2)^c m=\frac{m!\Gamma(2q-1)}{2\Gamma(2q+m)}$$

$$2q-2=2(n+1)-2=2n$$

$$c_m=\frac{m!\,2n!}{(2n+m+1)!\cdot 2}.$$

Then

$$a_m=2\pi b_m/c_m=\frac{2\pi b_m\cdot 2\cdot(2n+m+1)!}{m!\,2n!}.$$

Now, $\frac{(2n+m+1)!}{m!}$ is a polynomial of proper degree $2n+1$. Such polynomial is a linear combination of the following $2n+2$ polynomials:

$$C_0(m)=1$$

$$C_1(m)=m$$

$$C_2(m)=m(m-1)$$

$$C_3(m)=m(m-1)(m-2)$$

$$\vdots$$

$$C_{2n+1}(m)=m(m-1)\ \ldots\ (m-2n)$$

Thus we have

$$a_m=\frac{4\pi b_m}{2n!}\cdot\sum_{i=0}^{2n+1}B_i\cdot C_i(m).$$

(B_i constant, $B_{2n+1}\neq 0$); then

$$a_m r^m e^{im\theta}=\frac{4\pi}{2n!}\sum_{i=0}^{2n+1}B_i C_i(m)b_m r^m e^{im\theta}$$

and, if $r<1$

$$\sum_{m=0}^{\infty}a_m r^m e^{im\theta}=\sum_{m=0}^{\infty}\frac{4\pi}{2n!}\cdot\sum_{i=0}^{2n+1}B_i C_i(m)\cdot b_m r^m e^{im\theta}$$

$$=\frac{4\pi}{2n!}\sum_m\sum_i B_i\cdot C_i(m)b_m r^m e^{im\theta}$$

$$=\frac{4\pi}{2n!}\sum_i\sum_m B_i C_i(m)b_m r^m e^{im\theta}$$

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(since $r < 1$)

$$= \frac{4\pi}{2n!} \sum_i B_i \sum_{m=0}^{\infty} C_i(m) b_m r^m e^{im\theta}.$$

Now,

$$g^{(k)}(z) = \sum_{m=0}^{\infty} C_k(m) b_m r^{m-k} e^{i(m-k)\theta}$$

or

$$g^{(k)}(z) = z^{-k} \cdot \sum_m C_k(m) b_m r^m \cdot e^{im\theta}$$

or

$$z^k g^{(k)}(z) = \sum_m C_k(m) b_m r^m e^{im\theta}.$$

Hence

$$G(z) = \sum_{m=0}^{\infty} a_m r^m e^{im\theta}$$

$$= \frac{4\pi}{2n!} \sum_{i=0}^{2n+1} B_i \cdot g^{(i)}(z) \cdot z^i.$$

Now, with $A_i = \dfrac{4\pi B_i}{2n!}$ we have (1)$\cdot A_{2n+1} \neq 0$ because $B_{2n+1} \neq 0$.

We now turn to the case when q is *not* an integer.

COROLLARY 6: *Let* $n < q < n+1$. *Then* $G \in B_q$ *may be written*

$$G(z) = \sum_{i=0}^{2n+1} A_i \cdot z^i \cdot g^{(i)}(z),\ A_{2n+1} \neq 0$$

where not only is $g^{(n-1)} \epsilon \Lambda_*$, $g \epsilon A$ *but* $g^{(2n+1)} = 0\left|\dfrac{1}{(1-r)^q}\right.$.

(*We shall see shortly that* $g^{(n-1)} \epsilon \Lambda_* \Rightarrow g^{(2n+1)} = 0(1/(1-r)^{n+1}.)$

Note that we do *not* claim that this g is the same as the g associated with Λ.

PROOF: Choose β such that $q+\beta = n+1$. $G \epsilon B_q$. $G \epsilon B_{q+\beta} = B_{n+1}$ since $\lambda(z) \leq 1$, and $\beta > 0$. Apply Theorem 5, getting $G(z) = \sum_{i=0}^{2n+1} A_i z^i g^{(i)}(z)$. The proof that $g^{(2n+1)} = 0(1/(1-r)^q)$ depends on

LEMMA 7: *Say* $f \epsilon H(U)$,

$$|f(re^{i\theta})| < k/(1-r)^l.$$

Then

$$|f'(re^{i\theta})| < k'/(1-r)^{l+1}.$$

PROOF: Pick z, $r = |z|$, $r < p < 1$, $C_\rho = \{z \mid |z| = \rho\}$. Then

$$f'(z) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(\zeta)}{|\zeta - z|^2} d\zeta$$

$$|f'(z)| \leq \frac{k}{(1-\rho)^l} \cdot \frac{1}{2\pi} \cdot \int_{C_\rho} \frac{|d\zeta|}{|\zeta - z|^2}.$$

We must estimate the last integral

$$\int_{C_\rho} \frac{|d\zeta|}{|\zeta - z|^2} = \int_0^{2\pi} \frac{\rho d\theta}{|\rho e^{i\theta} - z|^2}$$

(where $\zeta = \rho e^{i\theta}$, $d\zeta = \rho i e^{i\theta} d\theta$, so $|d\zeta| = \rho d\theta$).

Now, let $\rho = (1+r)/2$. If we set $z^* = z/\rho$, $|z^*| = 2r/(1+r) < 1$, when $r < 1$. Then.

$$\int_0^{2\pi} \frac{\rho d\theta}{|\rho e^{i\theta} - z|^2} = \frac{1}{\rho} \int_0^{2\pi} \frac{d\theta}{\left| e^{i\theta} - \frac{z}{\rho} \right|^2}$$

$$= \frac{2}{1+r} \int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - z^*|^2}.$$

Let us look at the last integral.

$$2\pi = \int_0^{2\pi} \mathrm{Re} \frac{e^{it} + z^*}{e^{it} - z^*} dt \qquad \text{(Poisson Kernel)}$$

$$= \int_0^{2\pi} \mathrm{Re} \frac{(e^{it} + z^*)(\overline{e^{it} - z^*})}{|e^{it} - z^*|^2} dt$$

$$= \int_0^{2\pi} \mathrm{Re} \frac{1 - |z^*|^2 + (\overline{e^{it}} z^* - e^{it} \bar{z}^*)}{|e^{it} - z^*|^2} dt$$

Now $\overline{e^{it}} z^* - e^{it} \bar{z}^*$ is imaginary so

$$= \int_0^{2\pi} \frac{1 - |z^*|^2}{|e^{it} - z^*|^2} dt$$

$$= (1 - |z^*|^2) \int_0^{2\pi} \frac{dt}{|e^{it} - z^*|^2}$$

Thus

$$\frac{2}{1+r} \int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - z^*|^2} = \frac{2}{1+r} \cdot \frac{2\pi}{1 - |z^*|^2}$$

Now since $G \epsilon B_q$, $n < q < n+1$ we see that

(***) $$(1-r)^q \cdot z^i \cdot g^{(i)}(z) = 0(1)$$

for $i > 2n+1$.

$$G(z)\,(1-r)^q = 0(1) + (1-r)^q g^{2n+1'}(z) \cdot z^i$$

by (***). Thus $g^{2n+1} = 0(1/(1-r)^q)$.

This concludes the proof of *Corollary 6*.

We ask the following question: Is every sum of the form

$\Sigma\, A_i \cdot z^i \cdot g^{(i)}(z)$ in B_q ? First, if *q is an integer*, the answer is yes. For then $g^{n-1} \epsilon \Lambda_*$ which implies

$$\left.\begin{aligned} g^{n+1} &= 0(1/(1-r) \\ g^{n+2} &= 0(1/(1-r)^2) \\ \\ g^{2n+1} &= 0\left(\frac{1}{(1-r)^{n+1}}\right) = 0\left(\frac{1}{(1-r)^{q}}\right) \end{aligned}\right\} \quad \text{successively, by Lemma 7.}$$

Thus clearly

$$\sum_{i=0}^{2n+1} A_i \cdot z^i \cdot g^{(i)}(z) \in B_q.$$

If q is *not an integer*, then not every such sum is in B_q. Only those with $g^{2n+1} = 0(1/(1-r)^q)$ will be in B_q. $g \in \Lambda_*$ only implies that $g^{2n+1} = 0(1/(1-r)^{n+1})$, and $0(1/(1-r)^{n+1})$ is a weaker condition than $0(1/(1-r)^q)$, since $q < n+1$.

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