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Remarks on a Problem of Rademacher in the Theory of Modular Forms*

Marvin I. Knopp**

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Rademacher quite some time ago posed the question of deriving the classical functional equation of $1/\eta(z)$ ($\eta(z)$ is the Dedekind modular form) directly from the exact expression he had found for the partition function, $p(n)$, which arises as the Fourier coefficient of $1/\eta(z)$. Although he had been able to solve a similar problem for the absolute invariant $J(z)$, Rademacher was unable to solve the problem for $1/\eta(z)$. We here relate this question to some more recent results of Douglas Niebur, which reduce the problem to one of identically zero Poincaré series of degree $-5/2$.

Key words: Dedekind function; modular form; partition function; Poincaré series.

1. In [6][1] Rademacher introduced a method for recapturing the functional equation

$$J(-1/z)=J(z)$$

of the well-known modular invarient, $J(z)$, directly from the explicit representation of the Fourier coefficients of $J(z)$ that he had obtained earlier [5]. Subsequently, Rademacher quite naturally raised the question of finding an analogous method for deriving the functional equation of the modular form $\eta^{-1}(z)$ from the exact formula he had found for the partition function, $p(n)$, which is generated as the Fourier coefficient of $\eta(z)^{-1}$. It is the purpose of this note to consider the relationship of this problem to the results of Niebur [3]. We shall show that, in one sense at least, Niebur's results (which are quite far-reaching and deep) solve Rademacher's problem; in another sense the problem remains open. The apparent paradox will be explained in due course.

2. The *Dedekind function* $\eta(z)$ is defined by the infinite product.

$$\eta(z)=e^{\pi iz/12}\prod_{m=1}^{\infty}(1-e^{2\pi imz}),\qquad \operatorname{Im} z>0. \tag{1}$$

A simple combinatorial argument [1, pp. 32-34] shows that

$$\eta(z)^{-1}=e^{2\pi i(23/24)z}\left\{e^{-2\pi iz}+\sum_{n=0}^{\infty}p(n+1)e^{2\pi inz}\right\}, \tag{2}$$

where $p(n)$ is the number of unrestricted partitions of n into positive integers. Now it is obvious from either (1) or (2) that

$$\eta(z+1)=e^{\pi i/12}\,\eta(z), \tag{3}$$

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** Present address: The University of Illinois, Chicago, Illinois 60680
[1] Figures in brackets indicate the literature references at the end of this paper.

while it is a deep property of $\eta(z)$ that

$$\eta(-1/z) = e^{-\pi i/4} z^{1/2} \eta(z), \qquad \operatorname{Im} z > 0, \tag{4}$$

where $-\pi \leqslant \arg z < \pi$ is the convention adopted for calculating $z^{1/2}$ [1, pp. 41–43].

It follows from (1), (3), and (4) that $\eta(z)$ is a *modular cusp form of degree* $-1/2$; that is,

$$\eta(iy) \to 0, \text{ as } y \to +\infty, \tag{5}$$

and

$$\eta(Mz) = v_\eta(M)\ (cz+d)^{1/2}\, \eta(z), \tag{6}$$

for all $Mz = (az+b)\ /\ (cz+d)$, with a, b, c, d rational integers such that $ad - bc = 1$. Here $v_\eta(M)$ is a 24th root of unity, independent of z. The collection $\{v_\eta(M)\}$ is called the *multiplier system of* $\eta(z)$. It follows that $\eta(z)^{-1}$ is a modular form of degree 1/2, with multiplier system $\bar{v}_\eta$; $\eta(z)^{-1}$ has a *pole* at $i\infty$, in the sense that

$$|\ \eta(iy)^{-1}\ | \to +\infty, \text{ as } y \to +\infty. \tag{7}$$

Using (2) and (4) and a refined version of the "circle method," Rademacher [4] derived the exact formula

$$p(n) = \frac{e^{\pi 1/4}}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn}\left\{\frac{\sinh(\nu\sqrt{n-1/24}\,/k)}{\sqrt{n-1/24}}\right\}, \tag{8}$$

where $\nu = \pi\ \sqrt{2/3}$ and

$$A_k(n) = \sum_{\substack{h (\operatorname{mod} k)\\ (h,k)=1}} v_\eta(M_{k,-h}) \exp\left\{-\frac{2\pi i}{k}\left[\left(n - \frac{1}{24}\right)h + \frac{1}{24}h'\right]\right\}; \tag{9}$$

here $M_{k,-h} = \begin{pmatrix} h' & k' \\ k & -h \end{pmatrix}$, with $-hh' - kk' = 1$.

Rademacher's problem may be stated as follows. Suppose we regard the function $\eta(z)^{-1}$ as being *defined* by (2), (8), and (9), ignoring the definition (1) and the number-theoretic interpretation of $p(n)$ as the partition function. Show *from this definition* that $\eta(z)^{-1}$ is a modular form of degree 1/2 and multiplier system $\bar{v}_\eta$, that is, that

$$\eta(Mz)^{-1} = \bar{v}_\eta(M)(cz+d)^{-1/2}\eta(z)^{-1}, \tag{10}$$

with M as before. Since it is obvious from (2) that

$$\eta(z+1)^{-1} = e^{-\pi i/12}\eta(z)^{-1},$$

the problem reduces to showing that

$$\eta(-1/z)^{-1} = e^{\pi i/4} z^{-1/2} \eta(z)^{-1}, \qquad \operatorname{Im} z > 0. \tag{11}$$

3. The results of Niebur are valid for all automorphic forms of positive degree on H-groups, and, in particular, apply to the function $\eta(z)^{-1}$, as defined by (2) and (8). In this case Niebur's method

yields the following functional equation directly from the series (2), (8) [3, Theorem 3.2]:

$$e^{-\pi i/4}z^{1/2}\eta(-1/z)^{-1} - \eta(z)^{-1} = \left[\alpha \int_0^{i\infty} G(\tau)(\tau - \bar{z})^{1/2}d\tau\right]^{-},$$

where α is a nonzero constant, $[\beta]^{-}$ denotes the complex conjugate of β, the path of integration is the y axis, and $G(\tau)$ is the *Poincaré series* defined by

$$G(\tau) = \sum \frac{\exp\left\{\dfrac{2\pi i}{24}\dfrac{a\tau+b}{c\tau+d}\right\}}{v_\eta(M)(c\tau+d)^{5/2}}, \qquad \operatorname{Im} \tau > 0. \tag{13}$$

Here the sum is taken over all pairs of relatively prime integers c, d and a, b are rational integers chosen such that $ad-bc=1$; also, $M\tau = (a\tau+b)/(c\tau+d)$ and v_η is the multiplier system of $\eta(z)$.

Now (11) follows from (12) if it can be shown that

$$\int_0^{i\infty} G(\tau)\ (\tau - \bar{z})^{1/2}d\tau \equiv 0, \tag{14}$$

and Niebur has observed that (14) is equivalent to $G(\tau) \equiv 0$ [3, Theorem 4.1]. Thus Rademacher's problem is reduced to showing that $G(\tau) \equiv 0$. On the other hand, $G(\tau)$ is a modular cusp form of dimension $-5/2$ and multiplier system v_η (cf. the discussion in [2, pp. 272–280]), and we may show that such a modular form is necessarily $\equiv 0$. This can be done as follows. With $\eta(z)$ defined by (1), the function $f(z) = G(z)/\eta(z)$ is an *entire* modular form of degree -2 and multiplier system identically 1. That is,

$$f(Mz) = (cz+d)^2 f(z), \tag{15}$$

with M as in (6) and $\operatorname{Im} z > 0$, and

$$|f(iy)| \text{ has a finite limit as } y \to +\infty. \tag{16}$$

It is not too difficult to show that $f(z)$ satisfying (15) and (16) is identically 0 [2, p. 216]. Thus $G \equiv 0$ and Rademacher's problem is solved.

But only in a sense, for to maintain the spirit within which Rademacher proposed the problem, one should show that $G \equiv 0$ *directly from the expression* (13). This is as yet unresolved and apparently exceedingly difficult.

References

[1] Knopp, Marvin I., Modular Functions in Analytic Number Theory (Chicago, Markham Publishing Co., 1970).

[2] Lehner, J., Discontinuous groups and automorphic functions, Providence, American Mathematical Society, 1964.

[3] Niebur, Douglas, Automorphic integrals of arbitrary positive dimension and Poincaré series, Ph. D. Thesis, University of Wisconsin, 1968.

[4] Rademacher, Hans, A convergent series for the partition function $p(n)$, Proc. Nat. Acad. Sci. U.S.A. **23**, 78–84 (1937).

[5] Rademacher, Hans, The Fourier coefficients of the modular invariant $J(\tau)$, American Journal of Mathematics **60**, 501–512 (1938).

[6] Rademacher, Hans, The Fourier series and the functional equation of the absolute modular invariant $J(\tau)$, American Journal of Mathematics **61**, 237–248 (1939).

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