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An Application of Schur's Lemma on Irreducible Sets of Matrices in Continuum Mechanics

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In a recent study of the thermodynamic restrictions of a theory of compressible, viscoelastic fluids, Fong and Simmons (ZAMP **23,** No. 5 (1972)) encountered a problem of integrating the following matrix identity:

$$\underline{\underline{M}}[\,\underline{\underline{H}}\hat{\underline{\underline{U}}}_{,\underline{\underline{C}}}(\underline{\underline{H}}^T\underline{\underline{M}}\underline{\underline{H}})\underline{\underline{H}}^T-\hat{\underline{\underline{U}}}_{,\underline{\underline{C}}}(\underline{\underline{M}})]-\underline{\underline{H}}\hat{\underline{\underline{U}}}_{,\underline{\underline{C}}}(\underline{\underline{H}}^T\underline{\underline{H}})\underline{\underline{H}}^T\equiv\underline{\underline{O}},$$

where $\hat{U}_{,\boldsymbol{C}}$ denotes the gradient of the scalar-valued function $U=\hat{U}(\boldsymbol{C})$ with respect to its matrix argument $\boldsymbol{C}$ which is symmetric and positive-definite. The identity is valid for every symmetric positive-definite $\boldsymbol{M}$ and every unimodular $\boldsymbol{H}$. The symbol $\boldsymbol{H}^T$ denotes the transpose of the matrix $\boldsymbol{H}$. The solution of the problem is presented here in detail as an example of applying, probably for the first time, Schur's lemma on irreducible sets of matrices in theoretical continuum mechanics.

Key words: Continuum mechanics; elasticity; integration; matrix calculus; matrix identity; matrix theory; reducibility; Schur's lemma; strain energy.

1. Introduction

Continuum mechanics, or the mechanics of a deformable medium, depends heavily on the use of standard results in matrix theory for the formulation of problems and their solutions. For example, a "hyperelastic material" is characterized by the following constitutive equation when the thermal variables are ignored:

$$\underline{\underline{T}}=\rho\underline{\underline{F}}\,\underline{\underline{\sigma}}_{\underline{\underline{F}}}(\underline{\underline{F}})^T,\qquad T^k_m=\rho F^k_\alpha\frac{\partial\sigma}{\partial F^m_\alpha}. \tag{1.1}$$

Here $\boldsymbol{T}$ is the Cauchy stress tensor with a matrix representation T^k_m, which specifies the actual contact force per unit area in the spatial coordinate system x^k, $k=1, 2, 3$. The symbol ρ stands for the mass density per unit volume associated with a particle at x^k. To define $\boldsymbol{F}$, the deformation gradient tensor with a matrix representation F^k_α, we need to introduce a reference configuration κ with respect to which each material particle is given a coordinate label X^α, $\alpha=1, 2, 3$. The deformation gradient matrix is then defined as $F^k_\alpha=\partial x^k(X^\beta)/\partial X^\alpha$. The scalar function $\sigma(\boldsymbol{F})$ is called the strain-energy function of the hyperelastic material. The symbol $\sigma_{\boldsymbol{F}}$ stands for the gradient of σ with respect to $\boldsymbol{F}$, and is, therefore, itself a matrix with its transpose denoted by $\sigma_{\boldsymbol{F}}(\boldsymbol{F})^T$. Equation (1.1) states that the response of a hyperelastic material is completely determined for a given set of values of ρ and $\boldsymbol{F}$, provided the form of the scalar function σ can be determined experimentally. As it stands, σ depends on a 3×3 matrix variable or a total of nine components of the matrix F^k_α. A combination of physical requirement (strain energy must be frame-indifferent), and a standard result in matrix theory (polar decomposition of $\boldsymbol{F}$ into a product of an orthogonal $\boldsymbol{R}$ and a symmetric $\boldsymbol{U}$) reduces the

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number of variables in the function σ from nine to six, i.e., $\sigma(\boldsymbol{F}) = \sigma(\boldsymbol{U})$. For additional examples, see, e.g., Truesdell and Noll [1],[1] Murnaghan [2], etc.

In a recent study of the thermodynamic restrictions of a theory of compressible, viscoelastic fluids, Fong and Simmons [3] encountered the problem of integrating the following matrix identity:

$$\underline{\underline{M}}\ [\underline{\underline{H}}\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{H}}^T\underline{\underline{M}}\underline{\underline{H}})\underline{\underline{H}}^T - \hat{U}_{,\underline{\underline{C}}}(\underline{\underline{M}})] - \underline{\underline{H}}\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{H}}^T\underline{\underline{H}})\underline{\underline{H}}^T \equiv \underline{\underline{0}}\,, \tag{1.2}$$

where $\hat{U}_{,\boldsymbol{c}}$ denotes the gradient of a scalar-valued function $U = \hat{U}(\mathbf{C})$ with respect to its matrix argument $\boldsymbol{C}$ which is, by definition, symmetric and positive-definite.[2]. The identity is valid for every symmetric, positive-definite $\boldsymbol{M}$ and every unimodular $\boldsymbol{H}$, i.e., det $\boldsymbol{H} = 1$. It turns out that the integrability of (1.2) depends in a crucial way on two basic results in matrix theory. The purpose of this expository paper is to bring to the readers' attention these results which are well-known to mathematicians but not necessarily to workers in continuum mechanics:

Fact 1 The set of all symmetric, positive-definite matrices is irreducible.

Fact 2 If a matrix $\boldsymbol{Y}$ commutes with each matrix of an irreducible set, then $\boldsymbol{Y}$ is a scalar matrix, i.e., $\boldsymbol{Y} = \lambda \boldsymbol{I}$.

In section 2, the notion of "reducibility" of a set of matrices is first defined. A proof relating the notion of "reducibility" with that of an invariant subspace is also given. In section 3, we prove "fact 1" with a scheme of reasoning essentially due to Newman [4]. In section 4, we begin with Schur's lemma on irreducible sets of matrices and use it to prove "fact 2." The integration of (1.2) using both facts 1 and 2 is given in section 5. Finally, a discussion of the significance of the new result appears in section 6.

2. Reducibility of a Set of Matrices

We reproduce here the formal definition of the notion of "reducibility" of a set of matrices, $\mathscr{A} = \{\boldsymbol{A}_{(n\times n)}\}$, over the complex field, as presented by Newman [5]. The set $\mathscr{A}$ is said to be *reducible* if there exists a fixed, nonsingular matrix $\boldsymbol{S}_{(n\times n)}$ and fixed positive integers p, q, such that for each $\boldsymbol{A}$ in $\mathscr{A}$,

$$\underline{\underline{S}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{S}} = \begin{pmatrix} \underline{\underline{B}}_{(p\times p)} & \underline{\underline{O}} \\ \underline{\underline{D}}_{(q\times p)} & \underline{\underline{E}}_{(q\times q)} \end{pmatrix} \tag{2.1}$$

The symbol $\boldsymbol{O}$ represents a block of zeros with, of course, p rows and q columns. If no such $\boldsymbol{S}$ can be found, the set $\mathscr{A}$ is said to be *irreducible*. Examples of reducible and irreducible sets of matrices are:

Example 2.1 The set consisting of a single $n \times n$ matrix A alone, $n > 1$, is reducible.

Example 2.2 The set consisting of all 2×2 matrices of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ is reducible.

Example 2.3 The set consisting of all $n \times n$ matrices of the form $\begin{pmatrix} \underline{\underline{U}} & \underline{\underline{V}} \\ \underline{\underline{V}} & \underline{\underline{U}} \end{pmatrix}$ with respect to some fixed partitioning is reducible.

Example 2.4 The set of all $n \times n$ column stochastic matrices having all column sums equal to 1 is reducible.

Example 2.5 The set $\mathscr{A} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ is irreducible.

Additional examples of irreducible sets will be given in the next section. For those reducible sets given in the above examples, the reader can find the corresponding fixed matrix $\boldsymbol{S}$ in the book by

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[1] Figures in brackets indicate the literature references at the end of this paper.

[2] The scalar function $\hat{U}$, as it appears in reference [3], also depends on a scalar parameter ζ, i.e., $U = \hat{U}(\underline{\underline{C}}, \zeta)$. For our purposes here, this dependence is suppressed for brevity.

Newman [5]. We now wish to interpret the notion of reducibility by proving its equivalence to the existence of an invariant subspace:

Remark 2.1 Let V be the n-dimensional vector space over the complex field, and let $\mathscr{A}=\{\mathbf{A}_{(n\times n)}\}$ be a set of matrices which is reducible. Then there exists a subspace W in V such that W is invariant under any sequence of transformations given by the matrices in the set $\mathscr{A}$.

PROOF: Interpreting matrices in $\mathscr{A}$ as transformations, we have

$$\{\underline{\underline{A}}\,\underline{v}\}=\left\{\underline{\underline{S}}^{-1}\begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}\underline{\underline{S}}\,\underline{v}\right\}, \tag{2.2}$$

valid for each $\boldsymbol{v}$ in V and each $\boldsymbol{A}$ in the reducible set $\mathscr{A}$. Consider the set of vectors of the form

$$\underline{w}=\begin{pmatrix}\underline{0}_{(p\times 1)}\\ \underline{y}_{(q\times 1)}\end{pmatrix}. \tag{2.3}$$

Then we can define a subspace W consisting of all the $\boldsymbol{w}$'s with the property that W is invariant under matrix transformations of the type

$$\begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}, \qquad \text{i.e.,} \qquad \begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}\underline{w}=\underline{w}_1$$

is necessarily an element of W. We now wish to show that W is also invariant under the set $\mathscr{A}$. Let $\boldsymbol{S}^{-1}\boldsymbol{w}=\boldsymbol{u}$, i.e., $\boldsymbol{S}\,\boldsymbol{u}=\boldsymbol{w}$, and $\boldsymbol{S}\,\boldsymbol{u}_1=\boldsymbol{w}_1$. Then we have

$$\underline{\underline{S}}^{-1}\begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}\underline{\underline{S}}\,\underline{u}=\underline{\underline{S}}^{-1}\begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}\underline{w}=\underline{\underline{S}}^{-1}\underline{w}_1=\underline{u}_1.$$

Since a fixed transformation matrix $\boldsymbol{S}$ when applied to all the vectors in a subspace W does not alter the collection of vectors in that subspace, we conclude that the reducibility of $\mathscr{A}$ implies the existence of an invariant subspace W under $\mathscr{A}$.

Remark 2.2 Given a subspace W of the n-dimensional vector space V and given a set of matrices $\mathscr{A}=\{\boldsymbol{A}_{(n\times n)}\}$ under which W remains invariant, then the set $\mathscr{A}$ is reducible.

PROOF: Let the subspace W be of dimension q, $q<n$, and let $\boldsymbol{w}$ be any vector in W. Then there exists a linear transformation with matrix $\boldsymbol{S}$ such that every vector $\boldsymbol{w}$ can be brought to the form:

$$\underline{w}=\underline{\underline{S}}\begin{pmatrix}\underline{0}_{(p\times 1)}\\ \underline{y}_{(q\times 1)}\end{pmatrix}. \tag{2.4}$$

The condition that W is invariant under $\mathscr{A}$ implies $\{\boldsymbol{A}\boldsymbol{w}\}\subset\{\boldsymbol{w}\}$. Substituting the representation of $\boldsymbol{w}$ as given in (2.4) into the condition of invariance, we get

$$\left\{\underline{\underline{A}}\,\underline{\underline{S}}\begin{pmatrix}\underline{0}\\ \underline{y}\end{pmatrix}\right\}\subset\left\{\underline{\underline{S}}\begin{pmatrix}\underline{0}\\ \underline{y}\end{pmatrix}\right\},$$

$$\text{i.e.,}\left\{\underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}}\begin{pmatrix}\underline{0}\\ \underline{y}\end{pmatrix}\right\}\subset\left\{\begin{pmatrix}\underline{0}\\ \underline{y}\end{pmatrix}\right\}. \tag{2.5}$$

The statement (2.5) implies $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$ must be of the form $\begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}$. Hence the set $\mathscr{A}$ is reducible.

Combining remarks 2.1 and 2.2, we arrive at the following useful result:
Remark 2.3 A set of square matrices, $\mathscr{A}=\{\boldsymbol{A}_{(n\times n)}\}$, over the complex field is reducible, if, and only if, there exists an invariant subspace under $\mathscr{A}$.

3. Irreducibility of the Set of Symmetric, Positive-Definite Matrices

We now wish to use remark 2.3 to show whether a given set of matrices is reducible or not. The following remark is due to Newman [4]:
Remark 3.1 The set of matrices consisting of a diagonal matrix with nonzero and distinct eigenvalues, i.e., diag $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i \neq \lambda_j$ for $i \neq j$, $1 \leq i, j \leq n$, and a special matrix $\boldsymbol{J}=(J_{ij})$, with $J_{ij}=1$, $1 \leq i, j \leq n$, is irreducible.

PROOF: Let $\boldsymbol{x}=(x_1, x_2, \ldots, x_n)^T$ be a nonzero vector in the n-dimensional vector space V. Let $\boldsymbol{D}$ be the diagonal matrix with nonzero, distinct eigenvalues, $\lambda_1, \ldots, \lambda_n$. The proof for remark 3.1 can be broken into three steps as follows:
Step 1 For $\boldsymbol{x} \neq \boldsymbol{0}$, there exists a positive integer k such that $\boldsymbol{D}^k \boldsymbol{x} \neq \boldsymbol{0}$. Suppose the statement is false, i.e., $\boldsymbol{D}^i \boldsymbol{x}=\boldsymbol{0}$ for $1 \leq i \leq n$. Then it is possible to have the system:

$$\begin{cases} \lambda_1 x_1+\lambda_2 x_2+\ldots\ldots+\lambda_n x_n=0 \\ \lambda_1^2 x_1+\lambda_2^2 x_2+\ldots\ldots+\lambda_n^2 x_n=0 \\ \ldots\ldots\ldots \\ \lambda_1^n x_1+\lambda_2^n x_2+\ldots\ldots+\lambda_n^n x_n=0. \end{cases} \tag{3.1}$$

The system (3.1) has a nontrivial solution based on the hypothesis that $\boldsymbol{x} \neq \boldsymbol{0}$. Hence the determinant must vanish, contrary to the well-known result that a determinant of the form:

$$\Delta=\begin{pmatrix} \lambda_1 \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 \lambda_2^2 & \ldots & \lambda_n^2 \\ & & \\ \ldots & \ldots & \ldots \\ \lambda_1^n \lambda_2^n & \ldots & \lambda_n^n \end{pmatrix}=\lambda_1\lambda_2 \ldots \lambda_n \prod_{1\leq i<j\leq n} (\lambda_j-\lambda_i) \neq 0, \qquad (\lambda_i \neq \lambda_j \neq 0,\ i \neq j)$$

is never zero. Hence for $\boldsymbol{x} \neq \boldsymbol{0}$, there exists k such that

$$\lambda_1^k x_1+\lambda_2^k x_2+\ldots+\lambda_n^k x_n \neq 0.$$

Step 2 We now calculate the vector $\boldsymbol{J}\boldsymbol{D}^k\boldsymbol{x}$ and conclude that it is equivalent to the vector $\boldsymbol{y}=(1, 1, \ldots, 1)^T$ up to a nonzero scalar multiplying constant. Let us now calculate $\boldsymbol{D}\boldsymbol{y}$, $\boldsymbol{D}^2\boldsymbol{y}$, $\ldots$, $\boldsymbol{D}^{n-1}\boldsymbol{y}$, and obtain the following set of vectors:

$$\begin{aligned} \boldsymbol{y}&=(1, 1, \ldots, 1)^T, \\ \boldsymbol{D}\boldsymbol{y}&=(\lambda_1, \lambda_2, \ldots, \lambda_n)^T, \\ \boldsymbol{D}^2\boldsymbol{y}&=(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2)^T, \\ &\ldots\ldots \\ \boldsymbol{D}^{n-1}\boldsymbol{y}&=(\lambda_1^{n-1}, \lambda_2^{n-1}, \ldots, \lambda_n^{n-1})^T. \end{aligned}$$

Since the determinant is the well-known Vandermondian which is nonzero as long as the λ_i are distinct, we conclude that the above form a linearly independent set of n vectors and span the space.
Step 3 Since the set $\mathscr{A}=\{\boldsymbol{D}, \boldsymbol{J}\}$ of matrices when applied to a nonzero vector $\boldsymbol{x}$ generates the entire space, there is no proper subspace invariant under $\mathscr{A}$. Hence, by remark 2.3, the set $\mathscr{A}$ is irreducible.

Since both matrices $\boldsymbol{D}$ and $\boldsymbol{J}$ as defined in remark 3.1 are symmetric, it is trivial to conclude that,

Remark 3.2 The set of all symmetric matrices of any order over the complex field is irreducible.

Since any symmetric matrix can be made into a positive-definite symmetric one by the addition of a scalar matrix $\alpha\boldsymbol{I}$, where α is any nonzero scalar, and $\boldsymbol{I}$ is the identity matrix, and it is easy to show that such an addition does not affect the property of reducibility of a given set of symmetric matrices, we conclude that,

Remark 3.3 The set of all symmetric, positive-definite matrices of any order over the complex field is irreducibile. (This was stated earlier as "*Fact 1.*")

4. Shur's Lemma on Irreducible Sets of Matrices

We reproduce here the celebrated Schur's Lemma as stated in [5], p. 3:

THEOREM (Schur's Lemma): *Let* $\mathscr{A}=\{\boldsymbol{A}\}$, $\mathscr{B}=\{\boldsymbol{B}\}$ *be irreducible sets of* $\mathrm{n}\times\mathrm{n}$ *matrices,* $\mathrm{m}\times\mathrm{m}$ *matrices respectively. Let* $\boldsymbol{M}$ *be a fixed* $\mathrm{m}\times\mathrm{n}$ *matrix which determines a* $1-1$ *correspondence between* $\mathscr{A}$ *and* $\mathscr{B}$ *such that* $\boldsymbol{MA}=\boldsymbol{BM}$. *Then either* $\boldsymbol{M}=\boldsymbol{0}$, *or* $\mathrm{m}=\mathrm{n}$ *and* $\boldsymbol{M}$ *is nonsingular.*

The proof for the above theorem is given in [5] and is omitted here for brevity. It is, however, instructive to repeat the proof for an important corollary as follows:

COROLLARY: *If a matrix* $\boldsymbol{Y}$ *commutes with each matrix of an irreducible set* $\mathscr{A}$, *then* $\boldsymbol{Y}$ *is a scalar matrix, i.e.,* $\boldsymbol{Y}=\lambda\boldsymbol{I}$.

PROOF: Let λ be any eigenvalue of $\boldsymbol{Y}$. Then $\boldsymbol{Y}-\lambda\boldsymbol{I}$ is singular. It is easy to see that the matrix $\boldsymbol{Y}-\lambda\boldsymbol{I}$ also commutes with each matrix of $\mathscr{A}$. Schur's lemma now implies that $\boldsymbol{Y}-\lambda\boldsymbol{I}$ must be $\boldsymbol{0}$. Hence $\boldsymbol{Y}=\lambda\boldsymbol{I}$.

This corollary was referred to earlier in the introduction as "*Fact 2.*"

5. Integration of the Matrix Identity (1.2)

Let us rewrite (1.2) by introducing $\boldsymbol{Y}=\boldsymbol{H}\hat{U}_{,\underline{\underline{C}}}(\boldsymbol{H}^T\boldsymbol{H})\boldsymbol{H}^T$:

$$\underline{\underline{H}}\,\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{H}}^T\underline{\underline{M}}\,\underline{\underline{H}})\underline{\underline{H}}^T-\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{M}})=\underline{\underline{M}}^{-1}\underline{\underline{Y}}. \tag{5.1}$$

Since $\hat{U}$ depends on a symmetric argument, the gradient $\hat{U}_{,\boldsymbol{C}}$ is necessarily symmetric. This implies $\boldsymbol{Y}$ is symmetric as well as the left-hand side of (5.1). Since $\boldsymbol{M}$ is symmetric and positive-definite, $\boldsymbol{M}^{-1}$ is also symmetric and positive-definite. The identity (5.1) tells us that the product of two symmetric matrices, $\boldsymbol{M}^{-1}$ and $\boldsymbol{Y}$ is also symmetric. A standard result in matrix theory says that the necessary and sufficient condition for the product of two symmetric matrices to be again symmetric is that the two matrices must commute. Hence $\boldsymbol{M}^{-1}\boldsymbol{Y}=\boldsymbol{Y}\boldsymbol{M}^{-1}$. Since (5.1) is true for every positive-definite, symmetric $\boldsymbol{M}$, and since the set of all positive-definite, symmetric matrices is irreducible (Fact 1), the matrix $\boldsymbol{Y}$ must be a scalar. This completes the application of the corollary to Schur's lemma as stated in the last section (Fact 2).

With the matrix $\boldsymbol{Y}$ assuming the form of a scalar matrix $\lambda(\boldsymbol{H})\boldsymbol{I}$, we can write every term in (5.1) in the form of a total differential, i.e.,

$$\operatorname{tr}\{\underline{\underline{H}}\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{H}}^T\underline{\underline{M}}\,\underline{\underline{H}})\underline{\underline{H}}^T d\underline{\underline{M}}\}=d(\hat{U}(\underline{\underline{H}}^T\underline{\underline{M}}\,\underline{\underline{H}})), \tag{5.2}$$

$$\operatorname{tr}\{\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{M}})\qquad d\underline{\underline{M}}\}=d(\hat{U}(\underline{\underline{M}})), \tag{5.3}$$

$$\operatorname{tr}\{\lambda(\underline{\underline{H}})\underline{\underline{M}}^{-1}\qquad d\underline{\underline{M}}\}=d(\lambda(\underline{\underline{H}})\ln(\det\underline{\underline{M}})). \tag{5.4}$$

For an exposition of the use of matrix notation in the calculus of differentiable functions whose arguments are square matrices, see e.g., [6]. For our purposes here, it is sufficient to list the following formulas with which (5.2), (5.3), and (5.4) are derived:

$$\text{Given } \epsilon = \hat{\epsilon}(\underline{\underline{A}}) \qquad \text{, we have } d\hat{\epsilon} = \mathrm{tr}\,[\hat{\epsilon},_{\underline{\underline{A}}}^{T} d\underline{\underline{A}}]. \tag{5.5}$$

$$\text{Given } \hat{\epsilon}(\underline{\underline{A}}) = \det \underline{\underline{A}} \qquad \text{, we have } \hat{\epsilon},_{\underline{\underline{A}}} = (\det \underline{\underline{A}})(\underline{\underline{A}}^{-1})^{T}. \tag{5.6}$$

$$\text{Given } \hat{\epsilon}(\underline{\underline{A}}) = \hat{\phi}\,(\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{\underline{D}}), \text{ we have } \hat{\epsilon},_{\underline{\underline{A}}} = \underline{\underline{B}}^{T} \hat{\phi},_{(\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{\underline{D}})} \underline{\underline{D}}^{T}. \tag{5.7}$$

The identity (5.1) can now be integrated without difficulty:

$$\hat{U}(\underline{\underline{H}}^{T}\underline{\underline{M}}\,\underline{\underline{H}}) - \hat{U}(\underline{\underline{M}}) = \lambda(\underline{\underline{H}}) \ln (\det \underline{\underline{M}}) + \beta(\underline{\underline{H}}). \tag{5.8}$$

Detailed steps for the determination of the scalar functions $\lambda(\boldsymbol{H})$ and $\beta(\boldsymbol{H})$, are given in [3]. It is the main purpose of this expository article to demonstrate that without the mathematical results in the theory of irreducible sets of matrices, an identity on the scalar function $\hat{U}$ in the form (5.8) would not have been derived.

6. Significance of New Result Based on Identity (5.8)

An important contribution an applied mathematician can make in the field of science and engineering is to reduce the total number of variables in a given problem through a series of rigorous arguments, each of which can be further examined for its consistency with experiments. An example of this was given in the introduction of this paper where the strain energy function depends on six components of a symmetric matrix $\boldsymbol{U}$ instead of the nine components of the matrix $\boldsymbol{F}$. It is not surprising to many mathematicians that further simplification is possible by having the strain energy function to depend only on the three principal invariants of the symmetric matrix $\boldsymbol{U}$. The physical basis for the reduction of the number of variables from six to three is known as the condition of isotropy, where the hyperelastic material responds to an arbitrary deformation with no preference to its own orientation in an undistorted state.[3] A rigorous characterization of an isotropic, hyperelastic material requires the experimental determination of a strain energy function, say, $W = \hat{W}(L_1, L_2, L_3)$, where L_1, L_2, L_3 are some special combinations of the eigenvalues of the symmetric matrix $\boldsymbol{U}$. Recently Penn [7] reported the results of a series of experiments on the deformation of a peroxide vulcanized, pure-gum, natural rubber. He concluded from his experiments that the strain energy function, in general, cannot be separated as a sum of two parts:

$$W = \hat{W}(L_1, L_2, L_3) \neq F(L_1, L_2) + G(L_3). \tag{6.1}$$

In attempting to explain this experimental result, Fong and Simmons [3] studied the thermodynamic restrictions of a theory due to Bernstein, Kearsley, and Zapas [8, 9]. The theory was motivated by that of hyperelastic materials by replacing, among other things, the strain energy function $\hat{W}$ with a more general, time-dependent function, $U = \hat{U}(\boldsymbol{C}(t, \tau), t - \tau)$, where t and τ denote, respectively, the present and some past time between $-\infty$ and t. An identity as given in (1.2) on the gradient of the function $\hat{U}$ was derived, and Schur's lemma was applied in arriving at the identity (5.8) on $\hat{U}$. The significance of (5.8), as discussed in [3] and [10], is best described in terms of a decomposition result based on (5.8):

$$\hat{U}(L_1, L_2, L_3, \zeta) = F(L_1, L_2, \zeta) + G(L_3, \zeta) + H(L_1, L_2, \zeta) \ln L_3. \tag{6.2}$$

[3] For a thorough treatment of the notion of isotropy, see, e.g., Truesdell & Noll [1].

Since the Bernstein-Kearsley-Zapas' theory is known to describe responses of hyperelastic materials for some special forms of $\hat{U}$, it is conceivable that Penn's data [7] can be explained with an analogous decomposition on the strain energy function $\hat{W}$:

$$W=\hat{W}(L_1, L_2, L_3)=F(L_1, L_2)+G(L_3)+H(L_1, L_2)\ln L_3. \tag{6.3}$$

Further significance of the reduction of the form of $\hat{W}$ as stated in (6.3) will appear in a forthcoming paper [10].

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