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Bessel Functions *I* and *J* of Complex Argument and Integer Order

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A computer program is described for calculating Bessel functions $J_n(z)$ and $I_n(z)$, for z complex, and n a nonnegative integer. The method used is that of backward recursion, with strict control of error, and optimum determination of the point at which to begin the recursion.

Key words: Bessel functions; backward recursion; error bounds; Miller algorithm; difference equation.

1. Method

Given a complex number z and a positive integer NB, BESLCI calculates either

$$I_n(z), \qquad n=0, 1, \ldots, NB-1$$

or

$$J_n(z), \qquad n=0, 1, \ldots, NB-1$$

using double-precision arithmetic. The method used is described in [1] [1] and is based on algorithms of Olver [2] and Miller [3], applied to the difference equation

$$y_{n-1}=\frac{2n}{z}y_n-\mathrm{SIGN}\cdot y_{n+1}, \tag{1}$$

where SIGN is $+1$ for J's, -1 for I's.

The program sets MAGZ$=[|z|]$, the integer part of $|z|$, $p_{\mathrm{MAGZ}}=0$, $p_{\mathrm{MAGZ}+1}=1$, and then successively calculates

$$p_{n+1}=\mathrm{SIGN}\cdot\left(\frac{2n}{z}p_n-p_{n-1}\right), \qquad n=\mathrm{MAGZ}+1,\ \mathrm{MAGZ}+2, \ldots. \tag{2}$$

The sequence is strictly increasing in magnitude [1, sec. 6]. The program takes N to be the least n such that $|p_n|$ exceeds a number TEST defined in section 2 and section 3. It then sets $y_N^{(N)}=0$, $y_{N-1}^{(N)}=1/p_N$, and recurs backward using (1). To the working accuracy, the computed sequence $y_0^{(N)}, y_1^{(N)}, \ldots$ is the recessive solution of (1) which satisfies the boundary condition $y_{\mathrm{MAGZ}}=1$. From this solution, the I's or J's are found by normalizing:

$$J_n(z)=y_n^{(N)}/\mu \qquad n=0, 1, \ldots, NB-1$$

$$I_n(z)=y_n^{(N)}/\mu \qquad n=0, 1, \ldots, NB-1$$

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[1] Figures in brackets indicate literature references at the end of this paper.

where

$$
\begin{aligned}
&\mu=e^{iz}\left(y_0^{(N)}+2\sum_{n=1}^{N}(-i)^n y_n^{(N)}\right) && J\text{'s, Im } z>0\\
&\mu=y_0^{(N)}+2\sum_{n=1}^{[N/2]} y_{2n}^{(N)} && J\text{'s, Im } z=0\\
&\mu=e^{-iz}\left(y_0^{(N)}+2\sum_{n=1}^{N} i^N y_n^{(N)}\right) && J\text{'s, Im } z<0\\
&\mu=e^{-z}\left(y_0^{(N)}+2\sum_{n=1}^{N} y_n^{(N)}\right) && I\text{'s, Re } z>0\\
&\mu=y_0^{(N)}+2\sum_{n=1}^{[N/2]}(-1)^n y_{2n}^{(N)} && I\text{'s, Re } z=0\\
&\mu=e^{z}\left(y_0^{(N)}+2\sum_{n=1}^{N}(-1)^n y_n^{(N)}\right) && I\text{'s, Re } z<0.
\end{aligned}
\tag{3}
$$

2. Error Bounds for the y_n

For $n >$ MAGZ, the truncation error in $y_n^{(N)}$ is

$$T_n^{(N)}=y_n-y_n^{(N)}=p_n\sum_{r=N}^{\infty}\frac{\mathrm{SIGN}^{N-r}}{p_r p_{r+1}},$$

see [2]; equations 5.01 and 5.02. This error is bounded by using the following
LEMMA: *For* n $>$ *MAGZ*, *let*

$$k_n=\frac{p_{n+1}}{p_n}=\frac{2n}{z}-\frac{p_{n-1}}{p_n},$$

and let

$$\lambda_n=\frac{n+1}{|z|}+\sqrt{\left(\frac{n+1}{|z|}\right)^2-1}.$$

Let $\rho_n = min\ (|k_n|, \lambda_n)$. *Then for* $m \geqslant n$, $|k_m| > \rho_n$.

The lemma, for real z, is lemma 2 of [1]. The proof for complex z is essentially the same.

The program insures that

$$\mathrm{TEST}_1 \geqslant \sqrt{2\cdot 10^{\mathrm{NSIG}} p_L p_{L+1}}, \tag{4}$$

where $L = \max\ (\mathrm{MAGZ}+1,\ NB-1)$, and NSIG is the maximum number of significant decimal digits in a double-precision variable on the computer being used. Then N' is the least n such that $|p_n| > \mathrm{TEST}_1$, and N is the least $n \geqslant N'$ such that

$$|p_n| > \mathrm{TEST}=\sqrt{\frac{\rho_{N'}}{\rho_{N'-1}^2}}\cdot \mathrm{TEST}_1. \tag{5}$$

In consequence of (4) and (5), the relative truncation error $|T_n^{(N)}/y_n|$ is less than $\frac{1}{2}\cdot 10^{-\text{NSIG}}$ for all n in the range MAGZ $< n \leqslant L$; see [1, sec. 5].

For $n \leqslant$ MAGZ, it may be impossible to bound the relative truncation error in the above manner, owing to loss of precision due to cancellation in (1). Experience indicates that this loss is negligible except when the magnitude of $y_n^{(N)}$ oscillates as $n =$ MAGZ, MAGZ $-1, \ldots, 0$ in the back-recursion (e.g., calculating J's, with Re $z \gg$ Im z). In this case, there will be about D decimals of precision in the values $y_n^{(N)}$, where D is the number of decimals in $J_{\text{MAGZ}}(z)$ $(I_{\text{MAGZ}}(z))$ which corresponds to NSIG significant figures in the same quantity [1, sec. 5].

3. Normalization and Error Bounds for μ

The equations (3) were chosen to keep cancellation under control. Now

$$J_n(z) = \frac{i^{-n}}{\pi}\int_0^{\pi} e^{iz\cos\theta}\cos(n\theta)\,d\theta$$

[4], 9.1.21. The integrand never exceeds $e^{|\text{Im } z|}$ in magnitude, so $|J_n(z)| \leqslant e^{|\text{Im } z|}$. Similarly, $|I_n(z)| \leqslant e^{|\text{Re } z|}$. Thus each term of the sums (3) has magnitude less than twice that of the whole sum. In fact, these bounds on the magnitudes of $J_n(z)$ and $I_n(z)$ are rather weak, and cancellation in (3) is less than this would indicate.

Besides bounding the truncation error of the algorithm, the program provides an estimated bound for the truncation error of the normalization sum, defined by eq (3). In the first of these equations, this error is

$$S^{(N)} = e^{iz}\left\{y_0 - y_0^{(N)} + 2\sum_{n=1}^{\infty}(y_n - y_n^{(N)})\right\}.$$

For $n \leqslant$ MAGZ, a bound for the error term $y_n - y_n^{(N)}$ is unavailable. For MAGZ $< n < N$, $|y_n - y_n^{(N)}| \leqslant p_n\rho_n/(\rho_n^2 - 1)$; see [1, sec. 5]. To avoid storing all the p_n, the program allows only for terms for which $n \geqslant N$. Here $y_n^{(N)} = 0$, and

$$y_n = p_n \sum_{r=n}^{\infty} \frac{1}{p_r p_{r+1}}.$$

Therefore,

$$|y_n^{(N)} - y_n| \leqslant \left|\frac{1}{p_{n+1}}\right| \left\{1 + \left|\frac{p_n}{p_{n+2}}\right| + \left|\frac{p_n}{p_{n+2}}\frac{p_{n+1}}{p_{n+3}}\right| + \ldots\right\}$$

$$\leqslant \frac{1}{|p_{n+1}|}\left|1 + \frac{1}{\rho_n^2} + \frac{1}{\rho_n^4} + \ldots\right| \leqslant \frac{\rho_{N'}^2}{(\rho_{N'}^2 - 1)\,|p_{n+1}|},$$

compare section 2 above. Now let

$$R^{(N)} = 2\sum_{n=N}^{\infty}|y_n|.$$

Then

$$R^{(N)} \leqslant \frac{2\rho_{N'}^2}{\rho_{N'}^2 - 1}\sum_{n=N}^{\infty}\left|\frac{1}{p_{n+1}}\right| \leqslant \frac{2\rho_{N'}^2}{(\rho_{N'}^2 - 1)|p_{N+1}|}\left(1 + \frac{1}{\rho_{N'}} + \frac{1}{\rho_{N'}^2} + \ldots\right)$$

$$\leqslant \frac{2\rho_{N'}^2}{(\rho_{N'}^2 - 1)(\rho_{N'} - 1)\,|p_N|}.$$

The program sets $\mathrm{TEST}_1 \geqslant 2 \cdot 10^{\mathrm{NSIG}}$. The normalization factor μ is $e^{-iz}/J_{\mathrm{MAGZ}}(z)$ [1, sec. 5], so

$$\left|\frac{R^{(N)}}{\mu}\right| = \left|J_{\mathrm{MAGZ}}(z)\right| \cdot \frac{R^{(N)}}{e^{\mathrm{Im}\, z}} \leqslant R^{(N)}$$

$$\leqslant \frac{2\rho_{N'}^2}{(\rho_{N'}+1)(\rho_{N'}-1)^2\,|\rho_N|} \leqslant \frac{2\rho_{N'}^2}{(\rho_{N'}+1)(\rho_{N'}-1)^2} \cdot \frac{1}{2} \cdot 10^{-\mathrm{NSIG}}. \tag{6}$$

The bound (6) holds for the first, third, fourth and sixth equations of (3); the derivations are the same. Similarly, the second and fifth equations yield

$$\left|\frac{R^{(N)}}{\mu}\right| \leqslant \frac{2\rho_{N'}^3}{(\rho_{N'}^2-1)^2} \cdot \frac{1}{2} \cdot 10^{-\mathrm{NSIG}}.$$

These bounds are rather weak, and the error $|S^{(N)}/\mu|$ turns out to be less than $\frac{1}{2} \cdot 10^{-\mathrm{NSIG}}$.

4. References

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