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On the Eigenvalues of A + B and AB

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Using the usual field of values and the angular field of values inclusion sets are found for the eigenvalues of sums and products of $n \times n$ complex matrices. For instance if the field of values of B does not contain 0 it is found that the quotient of the field of values of A by that of B contains the eigenvalues of AB^{-1}. Applications are made to the polar form (AB where A is unitary and B positive semidefinite) and to products AB with A hermitian and $B+B^*$ positive definite.

Key words: Angular field of values; eigenvalues; field of values; hermitian; inertia; positive semidefinite.

This paper contains some simple results connecting the location of the complex eigenvalues of the sum and product of two $n \times n$ matrices A, B with certain point sets assigned to A and B: the familiar field of values and the angular field.

If M is an $n \times n$ complex matrix, its field of values is $F(M) \equiv \{x^*Mx \mid x \text{ a complex } n\text{-vector}, x^*x=1\}$.

$F(M)$ is known to be a closed bounded convex set containing all the eigenvalues of M; $F(M)$ coincides with the convex closure of the eigenvalues if M is normal. Apart from $F(M)$ we introduce the angular field $W(M)$:

$$W(M) \equiv \{x^*Mx \mid x \neq 0 \text{ complex } n\text{-vector}\}.$$

$W(M)$ clearly contains $F(M)$ and is the union of $F(M)$ together with all open rays with origin 0 containing a point $\neq 0$ of $F(M)$. Hence $W(M)$ has one of the following shapes: (1) the whole complex plane; (2) an angular area (not necessarily containing 0) with vertex 0 and angle α, $0 < \alpha \leqslant \pi$; (3) a half line starting from (and not necessarily containing) 0; (4) a line through 0; and (5) the single point 0. Cases (3) and (4) occur when $M = e^{i\theta}H$ and $H \neq 0$ is hermitian. It is easy to see that $W(N^*MN) = W(M)$ if $\det N \neq 0$, $W(M^*) = \overline{W(M)}$, and $W(M^{-1}) = \overline{W(M)}$ if $\det M \neq 0$.

In order to state our results easily, we use the following notation. If S_1 and S_2 are sets of complex numbers, we denote by S_1+S_2, S_1S_2, S_1/S_2, respectively the set of all numbers of the form s_1+s_2, s_1s_2, s_1/s_2 for $s_i \in S_i$ (in case of S_1/S_2 we assume $0 \notin S_2$). Clearly, we have $F(A+B) \subseteq F(A)+F(B)$ and $W(A+B) \subseteq W(A)+W(B)$ since $x^*(A+B)x = x^*Ax + x^*Bx$.

Let us start with an obvious theorem concerning the eigenvalues of $A+B$.

THEOREM 1: *If* λ *is an eigenvalue of* $A+B$, *then* $\lambda \in F(A)+F(B)$.

Proof: $\lambda \in F(A+B) \subseteq F(A)+F(B)$.

We proceed to search for similar theorems holding for products of matrices. Unfortunately the corresponding inclusion $F(AB) \subseteq F(A)F(B)$ is not in general valid; a counterexample will appear in remark 3 to follow.

However, we may note that quotients of matrices behave nicely.

THEOREM 2: *Let* $0 \notin F(B)$. *If* λ *is an eigenvalue of* $B^{-1}A$ *or of* AB^{-1}, *then* $\lambda \in F(A)/F(B)$.

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* An invited paper. This is a minor revision of a paper that has been referred to many times but has been very difficult to obtain since it appeared as an NBS Report No. 1367 in 1951. A good deal of work on closely related topics has appeared since then. The reader may therefore be interested in a partial bibliography at the end of this paper that deals primarily with spectra of products and facts about inertia. The wide range of references given in each of these articles is also recommended. The bibliography was compiled by C. R. Johnson at NBS. Present address of the author: University of Tübingen, Germany.

PROOF: Since $0 \notin F(B)$, 0 is not an eigenvalue of B, so B is nonsingular. Let $B^{-1}Ax = \lambda x$, $\|x\|=1$. Then $Ax=\lambda Bx$, $x^*Ax=\lambda x^*Bx$, $x^*Bx \neq 0$; hence $\lambda \in F(A)/F(B)$. In the case of AB^{-1} we may refer to the well-known theorem stating that AB^{-1} and $B^{-1}A$ have the same eigenvalues, or we may repeat the proof given above using a left-hand eigensolution x^* instead of x.

From theorem 2 we deduce the corresponding theorem for AB under suitable additional conditions.

THEOREM 3: *Let* A *be arbitrary, and* B *hermitian and positive semidefinite. If* λ *is an eigenvalue of* AB, *then* $\lambda \in F(A)F(B)$.

COROLLARY 3′: *Let* A *be unitary with eigenvalues* $\alpha_1, \ldots, \alpha_n$; *let* B *be hermitian and positive semidefinite with eigenvalues* $\beta_1 \leq \ldots \leq \beta_n$ *and let* λ *be any eigenvalue of* AB *or* BA. *Then* $\beta_1 \leq |\lambda| \leq \beta_n$, *and if all the* α_i *are contained in an arc* Φ *of the unit circle of length* $\leqq \pi$, *then* $\arg \lambda \in \Phi$.

The statements on β_n and $\arg \lambda$ follow immediately from theorem 3 since $|\alpha_i|=1$ and $F(A)$ is the convex closure of the α_i. The statement on β_1 is trivial if $\beta_1=0$, and is proven by passing over to $(AB)^{-1}$ if $\beta_1 > 0$.

Proof of theorem 3: First let B be nonsingular, $B=C^{-1}$. Then $\lambda = a/c$ with $a \in F(A)$ and $c \in F(C)$ according to theorem 2. If β_1 and β_n are the minimum and maximum eigenvalues of B, then β_1^{-1} and β_n^{-1} are the maximum and minimum eigenvalues of the positive definite matrix C, so $F(C)$ is the interval $[\beta_n^{-1}, \beta_1^{-1}]$, hence $c^{-1} \in [\beta_1, \beta_n] = F(B)$ and $\lambda \in F(A)F(B)$. The case $\det B = 0$ may be reduced to the nonsingular case by replacing B by $B + tI$, $t > 0$, $t \to 0$.

REMARK 3″: In theorem 3, the additional condition on B must not be dropped. For instance if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we have $F(A) = F(B) = [-1, 1]$, hence $F(A)F(B) = [-1, 1]$ which does not contain the eigenvalues $\pm i$ of AB. However, theorem 3 holds for arbitrary matrices if stated in a modified form involving the angular field instead of the field of values.

THEOREM 4: *Let* $0 \notin W(B)$. *If* λ *is an eigenvalue of* AB *or of* BA, *then* $\lambda \in W(A)\, W(B)$.

PROOF: Since $0 \notin W(B)$, B is not singular, $B = C^{-1}$. By theorem 2 we have $\lambda = a/c$ with $a \in F(A) \subseteq W(A)$ and, for some normalized x, we have

$$c = x^*Cx = (Cx)^*B^*(Cx) = y^*B^*y,$$

$$\bar{c} = y^*By \in W(B),\ c \neq 0,\ c^{-1} = \bar{c}/|c|^2 \in W(B).$$

It is worthwhile noting that, in general, we do not have $W(AB) \subseteq W(A)\, W(B)$: for instance whenever A and B are hermitian matrices which do not commute, then AB is not hermitian and $W(AB)$ is not real though $W(A)\, W(B)$ is real.

Theorem 4 may be used to obtain information concerning the signs of the real parts of eigenvalues.

THEOREM 5: *Let* C = AB, *where* A *is hermitian and* F(B) *is contained in the interior of the right half plane. Let* c_+, c_0, c_- *denote the number of positive, vanishing, and negative real parts of the eigenvalues of* C, *and let* a_+, a_0, a_- *denote the number of positive, vanishing, and negative eigenvalues of* A. *Then* $c_+ = a_+$, $c_0 = a_0$, *and* $c_- = a_-$.

The assumption on B is equivalent to each of the following: $F(B)$ is contained in the right half plane; $B + B^*$ is positive definite; or $B=H+iG$, with G and H hermitian and H positive definite.

COROLLARY 5′: *Let* A *be hermitian and* B *hermitian and positive definite. Then* AB *has as many positive, vanishing and negative eigenvalues as* A.[1]

[1] This corollary is not believed to be new, though the author has not been able to find it in the literature. It may be proved alternatively by applying the law of inertia to the matrices A and $TAT^*=TABT^{-1}$, where $T^*T=B$. Conversely, the law of inertia is a special case of the corollary.

PROOF OF THEOREM 5: B is nonsingular since $0 \notin F(B)$. Let λ be an eigenvalue of AB. By theorem 4 we have $\lambda \in W(A)W(B)$.

(a) Let us first assume A to be positive definite. Then AB is nonsingular and if $\gamma \in W(A)W(B) = W(B)$, then Re $\gamma > 0$. This means $c_+ = n = a_+$, $c_0 = 0 = a_0$, $c_- = 0 = a_-$. The proof is similar in case A is negative definite.

(b) Let us now treat the case of an indefinite, nonsingular A. Again every eigenvalue of AB is nonzero, but now $W(A)W(B)$ is the union of two opposite angular domains. We change the given matrix B continuously into βI, where β is some fixed value contained in $F(B)$, by letting $B(t) = t\beta I + (1-t)B$, $0 \leqslant t \leqslant 1$. The assumptions of case (b) are satisfied for A and $B(t)$. Thus we know the eigenvalues of $AB(t)$ never pass through 0. On the other hand, they vary continuously with t and never leave the domain $W(B) \cup (-W(B))$. Hence $W(B)$ contains as many eigenvalues of $AB(0) = AB$ as of $AB(1) = \beta A$. This means $c_+ = a_+$. Similarly $c_- = a_-$. In addition we have $c_0 = 0 = a_0$.

(c) The last case to consider is $\det A = 0$. Without loss of generality we may assume A in diagonal form. Let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \ \det A_1 \neq 0,$$

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Here B_4 has a_0 rows and columns. We have

$$AB = \begin{bmatrix} A_1 B_1 & A_1 B_2 \\ 0 & 0 \end{bmatrix}.$$

Therefore AB has a_0 eigenvalues equal to 0; the remaining $n - a_0$ eigenvalues of AB are those of $A_1 B_1$. It is clear that $F(B_1) \subseteq F(B)$, so A_1 and B_1 fulfill the assumptions of theorem 5, case (a) or (b), hence $c_+ = a_+$, $c_- = a_-$. Also we have $c_0 = n - (c_+ + c_-) = a_0$ which completes the proof.

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