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A Catalog of Minimal Blocks*

Arthur M. Hobbs**

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In this paper we provide a catalog of the minimal blocks with 10 and fewer vertices, together with a discussion of the methods and theorems used to produce the catalog. In addition, we prove a theorem which is a strengthening of a similar theorem of Fleischner [2] on the structure of minimal blocks.

Key words: Blocks; combinatorics; examples of graphs; graph theory; minimal blocks; planar graphs; thickness of graphs; two-connected graphs.

1. Description of the Catalog

The majority of the definitions used here will be found in [3],[1] with the terms "point", "line", and "cycle" replaced by *vertex, edge*, and *circuit*. In particular, a *block* is a connected graph with no cut vertex, and a block is *minimal* if no spanning subgraph of the block with fewer edges is also a block. We consider the graph with one vertex and no edges (the *vertex graph*) and the graph with two vertices and a single edge joining them (the *link graph*) to be minimal blocks. To distinguish between a path p and the graph which contains exactly the edges and vertices of p, we denote the graph by $|p|$; for simplicity of terminology, we will refer to both p and $|p|$ as "paths." If p is a path, $F(p)$ denotes the first vertex of p, and $L(p)$ denotes the last vertex of p. The undirected edge joining vertices ρ and σ is denoted by (ρ, σ) or (σ, ρ) interchangeably.

Let $A_1, \ldots, A_k$ be k disjoint graphs which are paths, with $k \geqslant 2$, such that path A_i contains m_i vertices, for each $i \in \{1, \ldots, k\}$, with $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_k \geqslant 1$. Let α and β be vertices not in $\bigcup_{i=1}^{k} V(A_i)$. For each $i \in \{1, \ldots, k\}$, let a_i be a Hamiltonian path in A_i. Let the graph $P(m_1, m_2, \ldots, m_k)$ be the graph with $V(P) = \bigcup_{i=1}^{k} V(A_i) \cup \{\alpha, \beta\}$ and $X(P) = \bigcup_{i=1}^{k} [X(A_i) \cup \{(\alpha, F(a_i)), (L(a_i), \beta)\}]$ (where $X(G)$ denotes the edge set of graph G) such that $A_1, \ldots, A_k$ are subgraphs of P. We call $P(m_1, \ldots, m_k)$ a *partition graph*; clearly a partition graph with n vertices is completely determined up to isomorphism by a partition of the integer $n-2$. Further, it is clear that each partition graph is a minimal block.

In the sequence $m_1, \ldots, m_k$, if $m_{s+1} = m_{s+2} = \ldots = m_{s+r}$ for some integer s and integer $r \geqslant 2$, we may write $m_1, \ldots, m_k$ in the form $m_1, \ldots m_s, r \times m_{s+1}, m_{s+r+1}, \ldots, m_k$. For example, $P(3, 4 \times 2)$ is shown in figure 1. Using this notation, the catalog at the end of this paper gives all minimal blocks with 10 and fewer vertices. Above each of the drawings of a graph with 7 or more vertices is a sequence in parentheses. This sequence is the degree sequence of the associated graph, i.e., the sequence of degrees of the vertices of the graph in descending order. The

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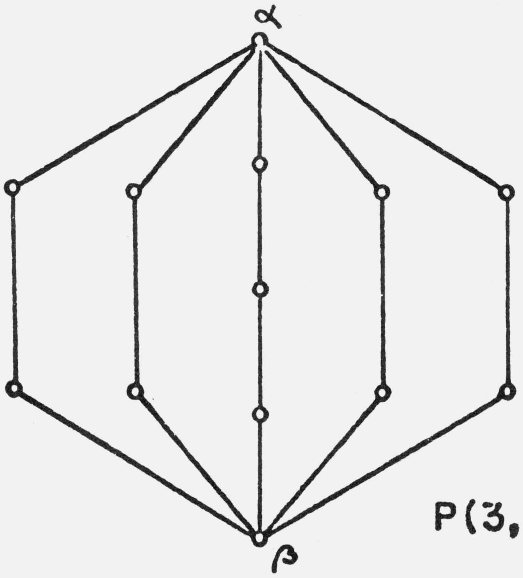
**Present address: Texas A and M University, College of Science, Department of Mathematics, College Station, Texas 77843.

[1] Figures in brackets indicate the literature references at the end of this paper.

minimal blocks presented in the catalog are all planar simply because both of the smallest nonplanar minimal blocks ($S(K_5)$ and $S(K_{3,3})$) have 15 vertices.



P(3, 4x2)

2. Theory and Method for Constructing the Catalog

A *suspended path* in a graph G is a path p of G such that all of the vertices in $V(p) - \{F(p), L(p)\}$ have degree 2 in G. The removal from and addition to a graph of a suspended path is similar to the removal from and addition to a graph of an edge. A graph G with blocks $B_1, \ldots, B_k$ is a *block-path chain* iff *either* $k = 1$ *or* $k > 1$, B_i and B_j have exactly one common vertex if $j \in \{i-1, i+1\}$, and B_i and B_j have no common vertices if $j \notin \{i-1, i, i+1\}$. Following [1], two vertices α and β in a graph G are *compatible* in G iff, for every path p with $F(p) = \alpha$ and $L(p) = \beta$, two vertices of p are joined by an edge of G iff the edge is in p.

Whitney [6] proved that in any block with three or more vertices there is an edge or a suspended path whose removal from the block results in a smaller block or a link graph. Thus a block with k vertices and m edges can be constructed from one with $k-1$ or fewer vertices or k vertices and $m-1$ edges by a process of adding an edge or a suspended path joining some distinct pair of vertices. If the block to be constructed is to be minimal, the removal of an edge from the constructed block cannot yield a block; thus each minimal block with k vertices can be constructed from a block with $k-1$ or fewer vertices by the addition of a suspended path joining some pair of distinct vertices. Further, it is easily shown that if G is a minimal block constructed from a block B by the addition of a suspended path joining some pair of distinct vertices of B, then B is a minimal block.

Dirac [1] has shown that removing an edge from a minimal block G results in a block-path chain in which any two nonadjacent cut vertices are compatible. Further, he has shown that if S is a block of this block-path chain, then either S is a link graph or the cut vertices in S of the block-path chain are not adjacent.[2] In particular, if a minimal block G is constructed from a block B by the addition of a suspended arc, then either B is a link graph or the two vertices in B of the suspended path are not adjacent in G and are compatible in B.

[2] Dirac also showed that any circuit in a minimal block contains a vertex of degree two. This fact can be used to give a very short proof of the theorem that a nonplanar minimal block has thickness two, where the thickness of a graph G is the smallest number of planar subgraphs of G whose union is G. See also Plummer [4] and Tutte [5].

Thus to construct all minimal blocks with k vertices, it is sufficient to find all minimal blocks with $k-1$ or fewer vertices, identify all nonadjacent compatible pairs of vertices in each of these minimal blocks, and join each such pair of vertices with a suspended path of length sufficient to produce a graph with k vertices. The resulting collection of graphs can be reduced by eliminating all but one graph in each isomorphism class of graphs.

The program described in the preceding paragraph is the one used for finding the catalog included in this paper. Some reductions in the labor of carrying out the program are easily found. It is easy to show that if the removal of two vertices from a minimal block results in a disconnected graph, then the vertices are compatible. Most compatible pairs of vertices can be detected simply by finding all such sets of two vertices. Further, the partition graphs with k vertices can be found immediately by finding all partitions of the integer $k-2$ into two or more parts. Since a partition graph can only be obtained from the link graph or from a smaller partition graph by the addition of a suspended path, it is easy to avoid producing any further copies of these particular graphs. Finally, it is clear that if there is an automorphism of a graph B which carries one pair of vertices of B into another pair of vertices of B, and if G (G') is formed from B by joining the first (second) pair of vertices by a suspended path of length m, then G and G' are isomorphic. Thus many isomorphic copies of graphs can be avoided by using only one compatible pair of vertices from a block B among all those pairs which can be carried into one another by automorphisms of B.

Once all minimal blocks with k vertices have been found, the job of eliminating the isomorphic copies produced in spite of the earlier tricks is considerably eased by first classifying the graphs by their degree sequences. Within each class, the isomorphic copies can then be eliminated with a small amount of additional labor.

3. Structure of Minimal Blocks

An *end block* of a block-path chain G is a block of G which contains at most one cut vertex of G. Following the terminology of [2], given a graph G, $D(G)$ is the set of all edges of G which are not incident with vertices whose degree in G is two. A *DT-subgraph* of a graph G is a subgraph H of G such that every edge of H is incident with a vertex whose degree *in* G is two. In [2], Fleischner proved a theorem which can be put in the following form:

THEOREM A: *Let* G *be a minimal block with* $|D(G)| \geqslant 2$, *and suppose* $\{\lambda_0, \lambda_1\} \subseteq D(G)$ *with* λ_1 *contained in an end block* B_0 *of* $G - \lambda_0$. *Then an end block of* $G - \lambda_1$ *is a subgraph of* B_0.

Using this result, Fleischner obtained an interesting theorem on the structure of minimal blocks. The following theorem is a strengthening of his theorem, using a modification of his proof.

THEOREM: *Let* G *be a finite minimal block with* $|D(G)| \geqslant 1$. Then either

(1) *there exists* $\lambda \in D(G)$ *such that both end blocks of* $G-\lambda$ *are* DT-*subgraphs of* G, *or*

(2) *there exists* $\{\lambda, \mu\} \subseteq D(G)$ *such that one end block* B_λ *of* $G - \lambda$ *and one end block* B_μ *of* $G - \mu$ *are* DT-*subgraphs of* G *and* $V(B_\lambda) \cap V(B_\mu)$ *contains at most one vertex.*

PROOF: If $|D(G)| = 1$, the theorem is trivial. Suppose $|D(G)| \geqslant 2$ and suppose that (1) does not hold, so that for every $\lambda \in D(G)$, one of the end blocks of $G-\lambda$ is not a DT-subgraph of G.

Let λ_0 be an element of $D(G)$ with end blocks $B_{1,0}$ and $B_{2,0}$ in $G-\lambda_0$ such that $B_{2,0}$ is not a DT-subgraph of G. For $i \in \{1, 2\}$, if $B_{i,0}$ is not a DT-subgraph of G, let $\lambda_{i,1} \in X(B_{i,0}) \cap D(G)$ (recall that $X(B)$ is the set of edges of graph B). By Theorem A, one end block $B_{i,1}$ of $G-\lambda_{i,1}$ is a subgraph of $B_{i,0}$. Continuing, for $i \in \{1, 2\}$, if $B_{i,j}$ is not a DT-subgraph of G, then $X(B_{i,j}) \cap D(G)$ is not empty; let $\lambda_{i,j+1} \in X(B_{i,j}) \cap D(G)$. Then by Theorem A, one end block $B_{i,j+1}$ of $G-\lambda_{i,j+1}$ is a subgraph of $B_{i,j}$. Since G is finite, we must eventually reach an edge $\lambda_{i,k} \in X(B_{i,k-1}) \cap D(G)$ such that one end block $B_{i,k}$ of $G-\lambda_{i,k}$ is a subgraph of $B_{i,k-1}$ and is a DT-subgraph of G. But since $B_{i,j}$ is a subgraph of $B_{i,j-1}$ for all $j \in \{1, \ldots, k\}$, $B_{i,k}$ is a subgraph of $B_{i,0}$.

If $B_{1,0}$ is a DT-subgraph of G, let $\lambda = \lambda_0$ and $B_\lambda = B_{1,0}$. If $B_{1,0}$ is not a DT-subgraph of G, by the above argument there exists $\lambda \in X(B_{1,0}) \cap D(G)$ such that an end block B_λ of $G - \lambda$ is a subgraph of $B_{1,0}$ and B_λ is a DT-subgraph of G. Also there exists $\mu \in X(B_{2,0}) \cap D(G)$ such that an end block B_μ of $G - \mu$ is a subgraph of $B_{2,0}$ and is a DT-subgraph of G. Then, since $V(B_{1,0}) \cap V(B_{2,0})$ contains at most one vertex, $V(B_\lambda) \cap V(B_\mu)$ also contains at most one vertex.

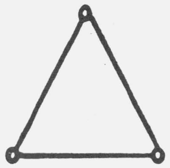
My thanks are due to Professor William T. Tutte and to Professor Crispin St. J. A. Nash-Williams for their help and support during the preparation of this paper.

4. Appendix. Catalog of Minimal Blocks With 10 and Fewer Vertices

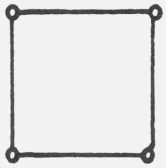
1 VERTEX

2 VERTICES

3 VERTICES



4 VERTICES



5 VERTICES

P(2, 1)

P(1, 1, 1)

6 VERTICES

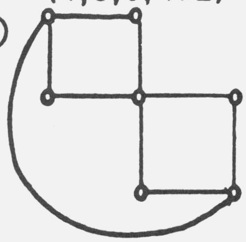
P(2, 2)

P(2, 1, 1)

P(1, 1, 1, 1)

7 VERTICES

① (4,3,3,4x2)



P(3,2)

P(3, 1, 1)

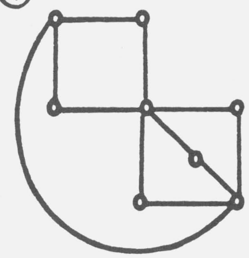
P(2,2,1)

P(2, 1, 1, 1)

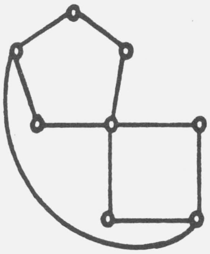
P(5x1)

8 VERTICES

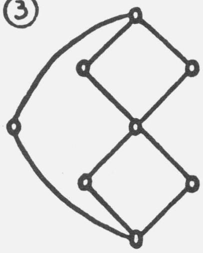
① (5,4,3,5x2)



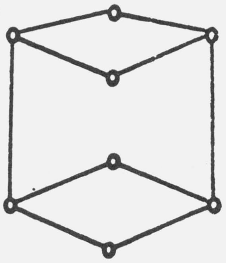
② (4,3,3,5x2)



③ (4,3, 3, 5x2)



④ (4x3, 4x2)



P(3, 3)

P(4, 1, 1)

P(3,2,1)

P(3, 1, 1, 1)

P(2,2,2)

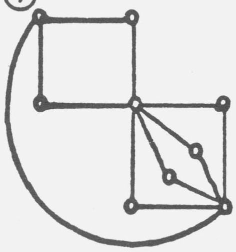
P(2, 2, 1, 1)

P(2, 4x1)

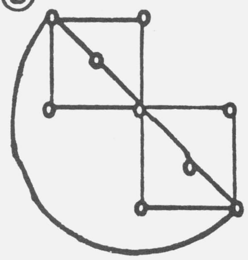
P(6x1)

9 VERTICES

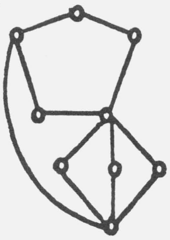
① (6,5,3,6x2)



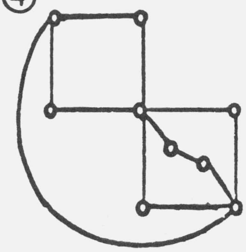
② (6,4,4, 6x2)



③ (5,4, 3,6x2)



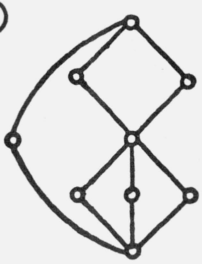
④ (5,4,3, 6x2)



9 VERTICES (CONTINUED)

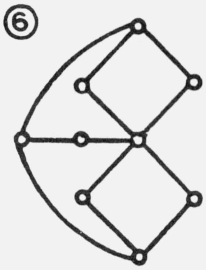
(5,4,3,6x2)

⑤



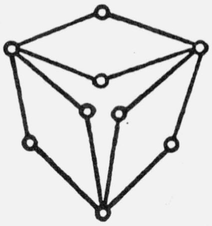
(5,3,3,3,5x2)

⑥



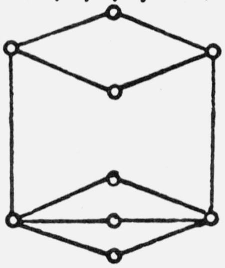
(4,4,4,6x2)

⑦



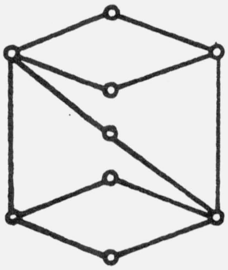
(4,4,3,3,5x2)

⑧



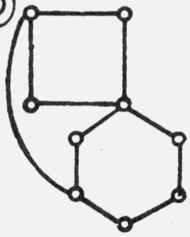
(4,4,3,3,5x2)

⑨



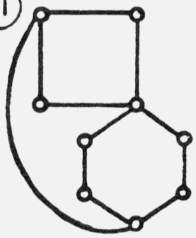
(4,3,3,6x2)

⑩



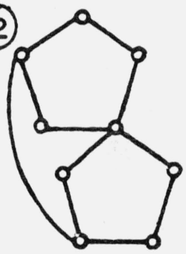
(4,3,3,6x2)

⑪



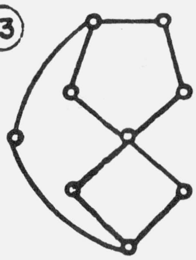
(4,3,3,6x2)

⑫



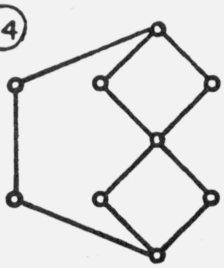
(4,3,3,6x2)

⑬



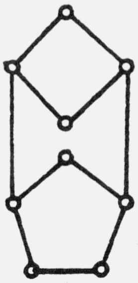
(4,3,3,6x2)

⑭



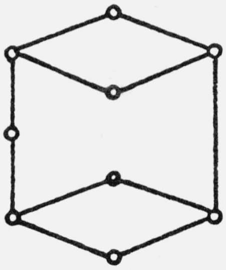
(4x3, 5x2)

⑮



(4x3, 5x2)

⑯



P(4,3)
P(5,1,1)
P(4,2,1)

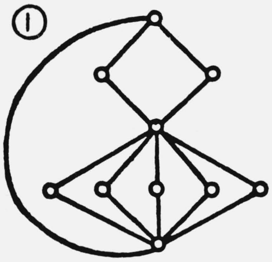
P(3,2,2)
P(3,3,1)
P(4,1,1,1)

P(3,2,1,1)
P(2,2,2,1)
P(3,4x1)

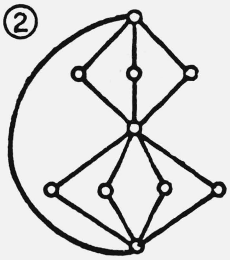
P(2,2,1,1,1)
P(2,5x1)
P(7x1)

10 VERTICES

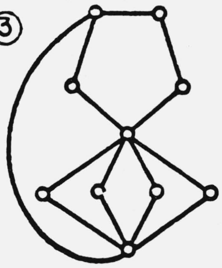
(1) (7,6,3,7x2)



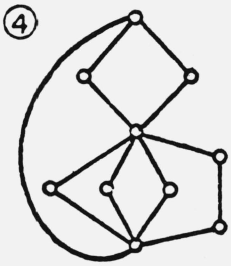
(2) (7,5,4,7x2)



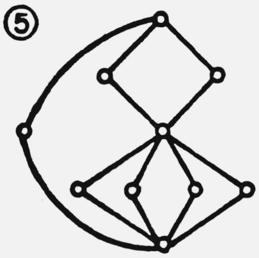
(3) (6,5,3,7x2)



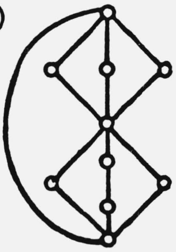
(4) (6,5,3,7x2)



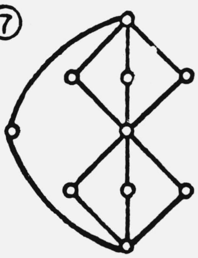
(5) (6,5,3,7x2)



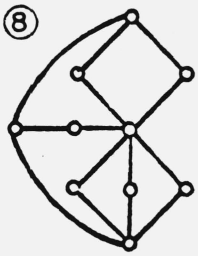
(6) (6,4,4,7x2)



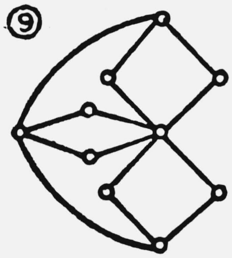
(7) (6,4,4,7x2)



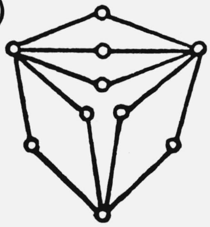
(8) (6,4,3,3,6x2)



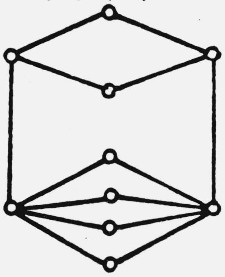
(9) (6,4,3,3,6x2)



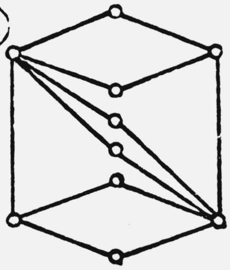
(10) (5,5,4,7x2)



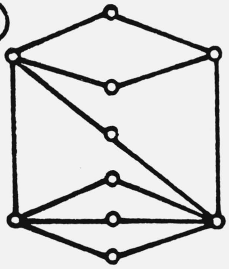
(11) (5,5,3,3,6x2)



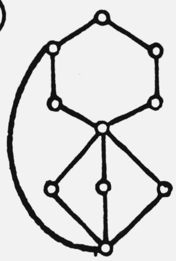
(12) (5,5,3,3,6x2)



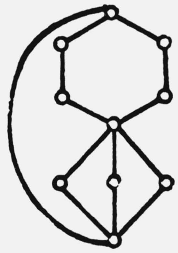
(13) (5,4,4,3,6x2)



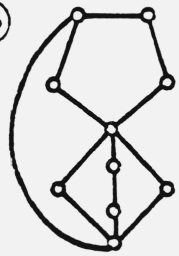
(14) (5,4,3,7x2)



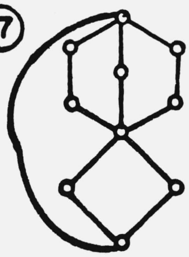
(15) (5,4,3,7x2)



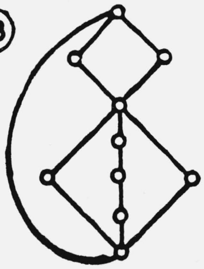
(16) (5,4,3,7x2)



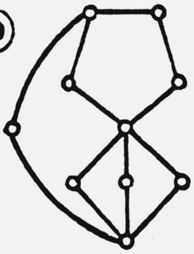
(17) (5,4,3,7x2)



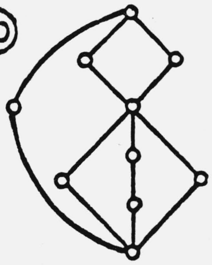
(18) (5,4,3,7x2)



(19) (5,4,3,7x2)

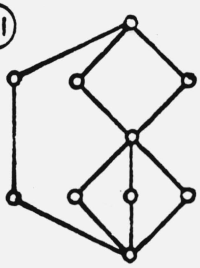


(20) (5,4,3,7x2)

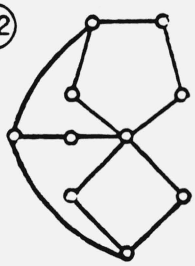


10 VERTICES (CONTINUED)

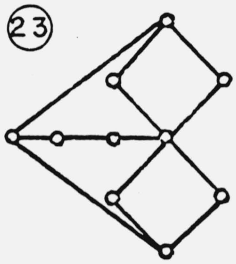
(21) (5,4,3,7x2)



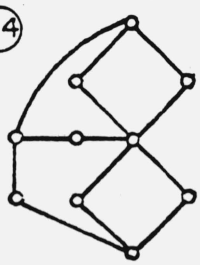
(22) (5,3,3,3,6x2)



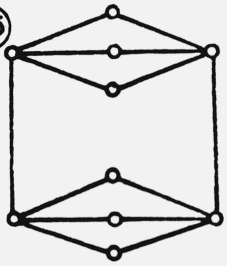
(23) (5,3,3,3,6x2)



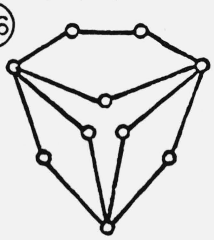
(24) (5,3,3,3,6x2)



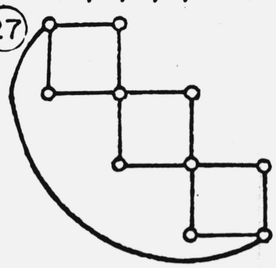
(25) (4x4,6x2)



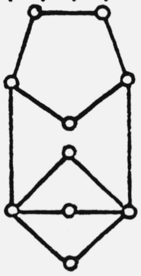
(26) (4,4,4,7x2)



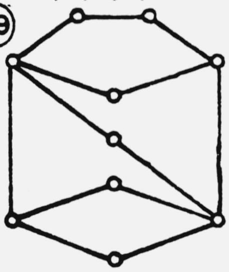
(27) (4,4,3,3,6x2)



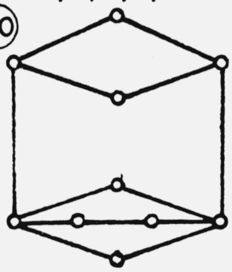
(28) (4,4,3,3,6x2)



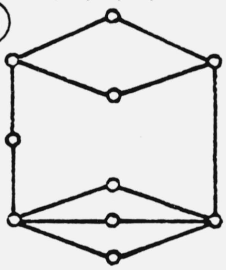
(29) (4,4,3,3,6x2)



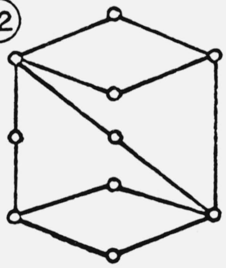
(30) (4,4,3,3,6x2)



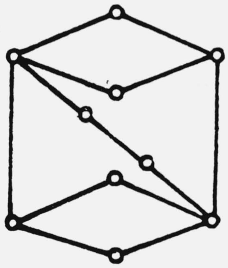
(31) (4,4,3,3,6x2)



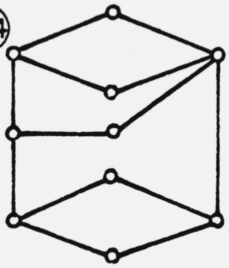
(32) (4,4,3,3,6x2)



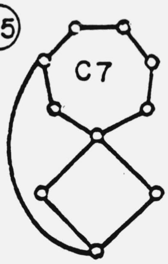
(33) (4,4,3,3,6x2)



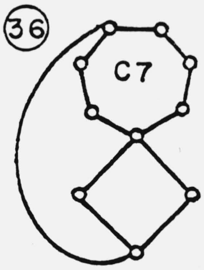
(34) (4,4x3,5x2)



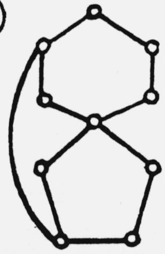
(35) (4,3,3,7x2)



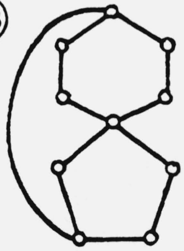
(36) (4,3,3,7x2)



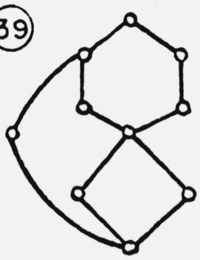
(37) (4,3,3,7x2)



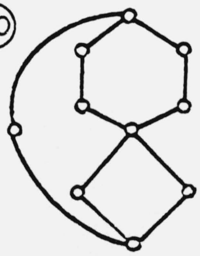
(38) (4,3,3,7x2)



(39) (4,3,3,7x2)



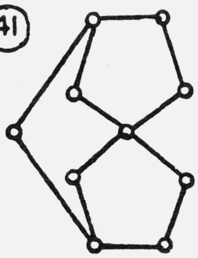
(40) (4,3,3,7x2)



10 VERTICES (CONTINUED)

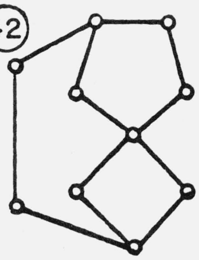
(4,3,3,7x2)

(41)



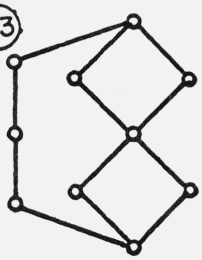
(4,3,3,7x2)

(42)



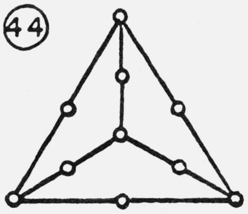
(4,3,3,7x2)

(43)



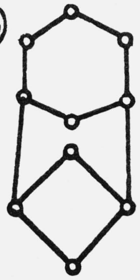
(4x3, 6x2)

(44)



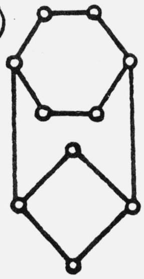
(4x3, 6x2)

(45)



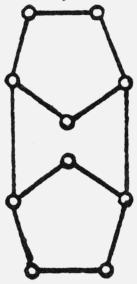
(4x3, 6x2)

(46)



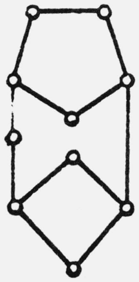
(4x3, 6x2)

(47)



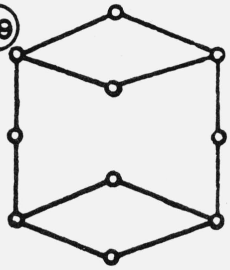
(4x3, 6x2)

(48)



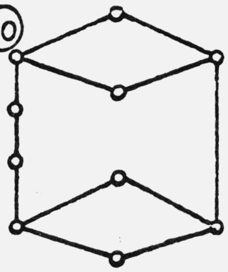
(4x3, 6x2)

(49)



(4x3, 6x2)

(50)



P(4,4)	P(4,3,1)	P(3,3,1,1)	P(3,5x1)
P(6,1,1)	P(3,3,2)	P(4x2)	P(2,2,4x1)
P(5,2,1)	P(5,1,1,1)	P(4,4x1)	P(2,6x1)
P(4,2,2)	P(4,2,1,1)	P(3,2,1,1,1)	P(8x1)
	P(3,2,2,1)	P(2,2,2,1,1)	

5. References

[1] Dirac, G. A., Minimally 2-connected graphs, J. Reine Angew. Math. **228** (1967), 204–216.

[2] Fleischner, H., The square of every two-connected graph is Hamiltonian, to appear.

[3] Harary, F., Graph Theory (Addison-Wesley Publishing Co., Reading, Mass., 1969).

[4] Plummer, M. D., On minimal blocks, Trans. Amer. Math. Soc. **134,** 85–94 (1968).

[5] Tutte, W. T., The thickness of a graph, Nederl. Akad. Wetensch. Indag. Math. **25,** 567–577 (1963).

[6] Whitney, H., Non-separable and planar graphs, Trans. Amer. Math. Soc. **34,** 339–362 (1932).

(Paper 77B1 & 2–378)