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Some Hamiltonian Results in Powers of Graphs*

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In this paper we show that the connectivity of the kth power of a graph of connectivity m is at least km if the kth power of the graph is not a complete graph. Also, we prove that removing as many as $k-2$ vertices from the kth power of a graph ($k \geqslant 3$) leaves a Hamiltonian graph, and that removing as many as $k-3$ vertices from the kth power of a graph ($k \geqslant 3$) leaves a Hamiltonian connected graph. Further, if every vertex of a graph has degree two or more, then the square of the graph contains a 2-factor. Finally, we show that the squares of certain Euler graphs are Hamiltonian.

Key words: Combinatorics; connectivity; Euler graphs; 2-factors; graph theory; Hamiltonian circuits; Hamiltonian connected; powers of graphs.

1. Introduction

We use the notation and terminology of [11][1], with the terms "point", "line", and "cycle" replaced by *vertex*, *edge*, and *circuit*. Further, we denote the set of edges of a graph G by $E(G)$. We follow the practice of representing a path by the sequence of vertices of the path. To distinguish between a path (circuit) p and the graph whose vertices and edges are exactly those of p, we denote the graph by $/p/$, and we call the graph a *pathoid (circuitoid).* We denote the distance between two vertices α and β in a graph G by $d_G(\alpha, \beta)$, and we denote the degree of a vertex α in G by $v_G(\alpha)$. The undirected edge joining vertices α and β is denoted by (α, β) or (β, α) interchangeably.

Given an integer $k \geqslant 1$, the kth *power* G^k of a graph G is a graph with $V(G^k) = V(G)$ and $(\alpha, \beta) \in E(G^k)$ iff $d_G(\alpha, \beta) \in \{1, 2, \ldots, k\}$. G^2 is called the *square* of G, and G^3 is called the *cube* of G. Given a path $p = \alpha_0, \alpha_1, \ldots, \alpha_k$, we let $F(p) = \alpha_0$, $L(p) = \alpha_k$, $V(p) = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$, and $I(p) = V(p) - \{F(p), L(p)\}$. We call p an $[\alpha_0, \alpha_k]$*-path* iff $F(p) = \alpha_0$ and $L(p) = \alpha_k$. Paths p and q are *internally disjoint* iff $I(p) \cap I(q)$ is empty. The number of elements in a set S is denoted by $|S|$.

2. Connectivity

Our first two theorems give useful information about a property of raising a graph to a kth power and the structure of a graph once the operation has been carried out. The first theorem is an easy consequence of the definition of power of a graph.

THEOREM 1: *Let* G *be a graph and let* $k = mn$, *where* m *and* n *are both positive integers. Then* $G^k = (G^m)^n$.

We will use this theorem to show that the next theorem is best possible. The *connectivity* $\kappa(G)$ of a graph G is the minimum over all pairs α, β of distinct vertices in G of the maximum number of distinct internally disjoint $[\alpha, \beta]$-paths in G. Since raising a graph G to the power k usually increases the number of edges present, it is not unreasonable to conjecture that G^k has higher connectivity than G has. Thus,

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[1] Figures in brackets indicate the literature references at the end of this paper.

THEOREM 2: *Let* G *be a finite graph, and let* k *be a positive integer. Then*

$$\kappa(G^k) \geqslant \min(|V(G)|-1, k\kappa(G)).$$

PROOF: If $\kappa(G)=0$ or $k=1$, the theorem is immediate. Similarly, if G^k is complete, the theorem holds because in that case, $\kappa(G^k)=|V(G)|-1$.

Now suppose G^k is not complete, $\kappa(G)>0$, and $k>1$. Let ξ and η be vertices of G which are not adjacent in G^k. Then by the definition of G^k, $d_G(\xi, \eta)>k$. Let $a_1, a_2, \ldots, a_{\kappa(G)}$ be $\kappa(G)$ internally disjoint $[\xi, \eta]$-paths in G, with $a_i=\xi, \gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,r_i}, \eta$. Since $d_G(\xi, \eta)>k$, $r_i \geqslant k$ whenever $i \in \{1, 2, \ldots, \kappa(G)\}$. Suppose $r_i=kt_i+s_i$, with $0 \leqslant s_i \leqslant k-1$. Then in G^k, ξ and η are joined by the paths

$$\xi, \gamma_{i,1}, \gamma_{i,k+1}, \gamma_{i,2k+1}, \ldots, \gamma_{i,t_ik+1}, \eta=p_{i,1},$$

$$\xi, \gamma_{i,2}, \gamma_{i,k+2}, \gamma_{i,2k+2}, \ldots, \gamma_{i,t_ik+2}, \eta=p_{i,2},$$

$$\ldots$$

$$\xi, \gamma_{i,s_i}, \gamma_{i,k+s_i}, \gamma_{i,2k+s_i}, \ldots, \gamma_{i,t_ik+s_i}, \eta=p_{i,s_i},$$

$$\xi, \gamma_{i,s_i+1}, \gamma_{i,k+s_i+1}, \gamma_{i,2k+s_i+1}, \ldots, \gamma_{i,(t_i-1)k+s_i+1}, \eta=p_{i,s_i+1},$$

$$\xi, \gamma_{i,s_i+2}, \gamma_{i,k+s_i+2}, \gamma_{i,2k+s_i+2}, \ldots, \gamma_{i,(t_i-1)k+s_i+2}, \eta=p_{i,s_i+2},$$

$$\ldots$$

$$\xi, \gamma_{i,k}, \gamma_{i,2k}, \ldots, \gamma_{i,(t_i-1)k}, \eta=p_k.$$

Each of these k paths contains only vertices of a_i and each path a_i contains vertices which can be used to form k internally disjoint $[\xi, \eta]$-paths in G^k in this way. Since the paths $a_1, \ldots, a_{\kappa(G)}$ are internally disjoint, between ξ and η in G^k there are $k\kappa(G)$ internally disjoint $[\xi, \eta]$-paths. Thus, G^k is $k\kappa(G)$-connected.

COROLLARY 2A: *If* G *is a finite graph with connectivity* $\kappa(G) \geqslant 1$, *if* k *is a positive integer, and if* $k\kappa(G) \geqslant |V(G)|-1$, *then* G^k *is a complete graph.*

Theorem 2 is best possible in the sense that additional conditions are needed to improve it, as the following examples show. If P is a pathoid with n vertices, if $\xi \in V(P)$ and has degree 1, and if r is a positive integer less than n, then ξ is of degree r in P^r. Thus, by Theorem 2, P^r has connectivity exactly r. Given positive integers k and m, and given $n \geqslant km+1$, let P be a pathoid of length $n-1$. Then $(P^m)^k=P^{mk}$ by Theorem 1, and so it is a graph of connectivity km which is the kth power of an m-connected graph. Finally, let C_n be a circuitoid with n vertices. Let $m>0$ and even, and let k be a positive integer and n be an integer no less than $km+1$; then $C_n^{(km)/2} = (C_n^{m/2})^k$ and is a minimally km-connected graph which is the kth power of a minimally m-connected graph.

3. *r*-Hamiltonian Powers of Graphs

Following [4] and [17], we call a graph G *Hamiltonian connected* iff every two distinct vertices of G are joined by a Hamiltonian path in G, and we call G *r-Hamiltonian* iff $|V(G)| \geqslant r+3$ and $G-\{\xi_1, \ldots, \xi_r\}$ is Hamiltonian for every set of r vertices $\{\xi_1, \ldots, \xi_r\} \subseteq V(G)$. Given a path p, we let $l(p)$ denote the length of p. Given disjoint paths $p=\xi_1, \xi_2, \ldots, \xi_r$ and $q=\eta_1, \eta_2, \ldots, \eta_s$, if ξ_r is adjacent in G to η_1, we denote the path $\xi_1, \ldots, \xi_r, \eta_1, \eta_2, \ldots, \eta_s$ by $(p), (q)$. Let ξ be a cut vertex of a connected graph G, and let $H_1, \ldots, H_r$ be all of the components of $G - \xi$. Then for each $j \in \{1, \ldots, r\}$, we call the subgraph

$$G - \bigcup_{\substack{i=1 \\ i \neq j}}^{r} V(H_i)$$

of G a *ξ-section* of G.

CONJECTURE 1: *If* G *is a finite connected graph,* k *is an integer no less than 3, and* $A \subseteq V(G)$ *such that* $|A| \leq k\kappa(G) - 3$, *then* $G^k - A$ *is Hamiltonian connected.*

CONJECTURE 2: *If* G *is a finite connected graph,* k *is an integer no less than 3, and* $r \leq$ *min* $(|V(G)| - 3, k\kappa(G) - 2)$, *then* G^k *is* r-*Hamiltonian.*

Because Theorem 2 is best possible and at least 2-connectedness is required for the presence of a Hamiltonian circuit in a graph, Conjecture 2 is best possible if it is true. Furthermore, if $0 \leq |V(G)| - 3 \leq k\kappa(G) - 2$, then $|V(G)| - 1 \leq k\kappa(G)$ and G^k is complete by Corollary 2A. In such a case, for any set $A \subseteq V(G)$ with $|A| \leq |V(G)| - 3$, $G^k - A$ is Hamiltonian. Thus we need only consider Conjecture 2 for the case in which $|V(G)| - 3 \geq k\kappa(G) - 2$ and $|A| \leq k\kappa(G) - 2$. We have not yet proven these conjectures in general, but in the next portion of this paper we give proofs for the case of $\kappa(G) = 1$.

Let G be a graph, k a positive integer, and A a subset of the vertices of G. Two distinct vertices ξ and η of G are *A-joined* iff there exists a path p in G joining ξ and η such that $I(p) \subseteq A$. Note that two adjacent vertices are *A-joined* for every set A of vertices in G. A path p of G is *(A, k)-solid* iff $\{F(p), L(p)\} \subseteq V(G) - A$, $|I(p) - A| \leq 2$, and $l(p) > k$.

LEMMA 1: *Let* $k \geq 3$ *be an integer,* G *a finite graph, and* $A \subseteq V(G)$ *with* $|A| \leq k - 3$. *Then* G *contains no* (A, k)-*solid paths.*

PROOF: This lemma is immediate from the definition of (A, k)-solid paths.

LEMMA 2: *Let* $k \geq 3$ *be an integer, let* G *be a graph, and let* $A \subseteq V(G)$ *such that* $|A| \leq k - 2$. *If* a *is a path of* G *such that* $|I(a) - A| \leq 1$, *then* $l(a) \leq k$.

PROOF: Since $|A| \leq k - 2$ and $|I(a) - A| \leq 1$, $|I(a)| \leq k - 1$. But $l(a) = |I(a)| + 1 \leq k - 1 + 1 = k$.

We next prove a very strong result that holds for special subsets A of vertices of a tree. Let ϕ be the empty set.

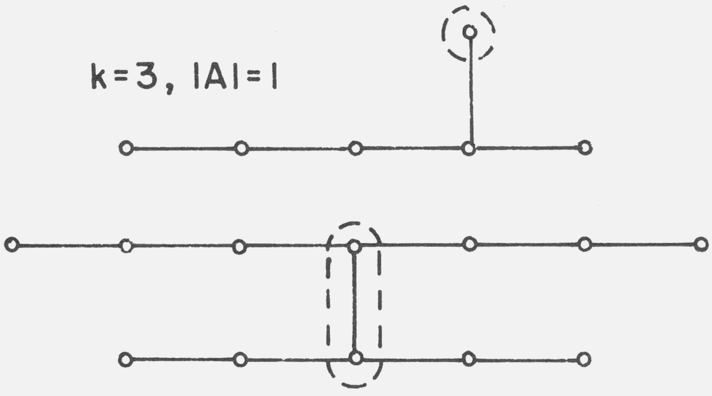
LEMMA 3: *Let* $k \geq 3$ *be an integer, let* T *be a finite tree, and let* $A \subseteq V(T)$. *Suppose no path in* T *is* (A, k)-*solid. If* ξ *and* η *are distinct* A-*joined vertices in* $V(T) - A$, *then* $T^k - A$ *contains a Hamiltonian* $[\xi, \eta]$-*path.*

PROOF: If $|V(T)| \leq k + 1$, then T^k is complete by Corollary 2A. Hence $T^k - A$ is a complete graph with at least the two vertices ξ and η, and so there is a Hamiltonian $[\xi, \eta]$-path in $T^k - A$.

Suppose the lemma is true for all trees T' with $|V(T')| < q$, and let T be a tree with q vertices which satisfies the conditions of the lemma. Let $T_1, T_2, \ldots, T_r$, $r \geq 1$, be the ξ-sections of T for which $V(T_i) - A \neq \phi$, and suppose $\eta \in V(T_r)$. Since a path which is (A, k)-solid in T_i, $1 \leq i \leq r$, is (A, k)-solid in T, the lemma holds for all subtrees T_i which have at least 2 vertices not in A. In tree T_i, $1 \leq i \leq r - 1$, let $\gamma_{i,1}$ be a vertex of $V(T_i) - A$ which is A-joined to ξ in T, and let $\gamma_{r,1} = \eta$, which is A-joined to ξ by assumption. In tree T_i, $1 \leq i \leq r$, if $V(T_i) - (A \cup \{\gamma_{i,1}\}) \neq \phi$, let $\gamma_{i,2}$ be a vertex of $V(T_i) - (A \cup \{\gamma_{i,1}\})$ which is A-joined to $\gamma_{i,1}$ in T, and if $V(T_i) - (A \cup \{\gamma_{i,1}\}) = \phi$, let $\gamma_{i,2} = \gamma_{i,1}$.

Since T is a tree, there is a unique $[\xi, \gamma_{1,2}]$-path a in T, and $\gamma_{1,1}$ is the only vertex in $I(a) - A$; by the definition of (A, k)-solid path, $\ell(a) \leq k$ since a is not (A, k)-solid. Thus $d_T(\xi, \gamma_{1,2}) \leq k$. Further, at most two vertices, ξ and $\gamma_{i,1}$, are in $I(s_i) - A$, where s_i is the unique $[\gamma_{i-1,1}, \gamma_{i,2}]$-path in T, $2 \leq i \leq r$; since T contains no (A, k)-solid path, s_i cannot have length greater than k, so $d_T(\gamma_{i-1,1}, \gamma_{i,2}) \leq k$ for $i \in \{2, \ldots, r\}$. Since $T_1, \ldots, T_r$ satisfy the conditions of this lemma and have fewer than q vertices each, in $T_i^k - A$ there is a Hamiltonian $[\gamma_{i,2}, \gamma_{i,1}]$-path a_i. Thus, $\xi, (a_1), (a_2), \ldots, (a_r)$ is a Hamiltonian $[\xi, \eta]$-path in $T^k - A$.

Examples of trees T and sets A which satisfy the conditions of Lemma 3 are shown in figure 1, where the vertices in the sets A are contained in dashed curves.



k=5, |A|=2

FIGURE 1

COROLLARY 3a: *Let* T *be a finite tree with at least 3 vertices, let* k *be an integer no less than 3, and let* $A \subseteq V(T)$ *such that no path of* T *is* (A, k)*-solid. Then, for each edge* $\lambda \in E(T-A)$, $T^k - A$ *contains a Hamiltonian circuit* c *with* $\lambda \in E(c)$.

THEOREM 3: *Let* T *be a finite tree, let* k *be an integer no less than* 3, *and let* $A \subseteq V(T)$. *Suppose* T *has no* (A, k)*-solid paths. Then* $T^k - A$ *is Hamiltonian connected.*

PROOF: The theorem is clearly true for a tree with 1 or 2 vertices, and it is vacuously true for a tree with no vertices. Suppose the theorem is true for every tree with fewer than q vertices, and let T be a tree with q vertices. By Lemma 3, we need only show that, for any two distinct vertices which are not A-joined in T, $T^k - A$ contains a Hamiltonian path joining them. Let ξ and η be two vertices of $V(T) - A$ which are not A-joined. Let a be the $[\xi, \eta]$-path in T and let $\alpha \in I(a) - A$. Let the components of $T - \alpha$ which contain vertices not in A be $C_1, \ldots, C_r$, with $\xi \in V(C_1)$ and $\eta \in V(C_r)$.

For $i \in \{2, 3, \ldots, r-1\}$, let ρ_i be a vertex of $C_i - A$ which is A-joined to α in T. Let $\rho_1 = \xi$. If there is a vertex of $C_r - A$ other than η which is A-joined to α, let ρ_r be such a vertex, and if η is the only vertex in $C_r - A$, let $\rho_r = \eta$. Otherwise, let ρ_r be a vertex of $C_r - A$ which is A-joined to η. For $i \in \{2, 3, \ldots, r-1\}$, let δ_i be a vertex of $C_i - A$ other than ρ_i which is A-joined to ρ_i (if no such vertex exists, let $\delta_i = \rho_i$). If there is a vertex of $C_1 - A$ other than ξ which is A-joined to α in T, let δ_1 be such a vertex, and if ξ is the only vertex in $C_1 - A$, let $\delta_1 = \xi$. Otherwise, let δ_1 be a vertex which is A-joined to ξ. Let $\delta_r = \eta$. By these choices, ξ is the only vertex not in A which can be on the path of T joining δ_1 and α, and η is the only vertex not in A which can be on the path of T joining α and ρ_r.

Each component C_i is a tree with fewer than q vertices. Further, since T contains no (A, k)-solid paths, C_i contains no (A, k)-solid paths, for each i. Thus the theorem holds for each tree C_i, and $C_i^k - A$ contains a Hamiltonian $[\rho_i, \delta_i]$-path p_i.

We note that in a tree S with no (A, k)-solid paths, if p is a path for which $\{F(p), L(p)\} \subseteq V(S) - A$ and $|I(p) - A| \leq 2$, then $l(p) \leq k$. Hence, $d_T(\delta_1, \rho_2) \leq k$ since at most ξ and α of $V(T) - A$ can be on the $[\delta_1, \rho_2]$-path of T. Also, $d_T(\delta_i, \rho_{i+1}) \leq k$ whenever $i \in \{2, 3, \ldots, r-2\}$, since at most ρ_i and α of $V(T) - A$ can be on the $[\delta_i, \rho_{i+1}]$-path in T. Further, $d_T(\delta_{r-1}, \alpha) \leq k$ since ρ_{r-1} is the only vertex not in A which can be on the $[\delta_{r-1}, \alpha]$-path of T, and $d_T(\alpha, \rho_r) \leq k$ since η is the only vertex not in A which can be on the $[\alpha, \rho_r]$-path in T. Therefore, $(p_1), (p_2), \ldots, (p_{r-1}), \alpha, (p_r)$ is a Hamiltonian $[\xi, \eta]$-path in $T^k - A$.

COROLLARY 3A: *Let* G *be a connected finite graph, let* k *be an integer no less than 3, and let* $A \subseteq V(G)$ *with* $|A| \leq k-3$. *Then* $G^k - A$ *is Hamiltonian connected.*

PROOF: Let T be a spanning tree in G. Since $|A| \leq k-3$, T contains no (A, k)-solid paths by Lemma 1. Therefore, $T^k - A$ is Hamiltonian connected by Theorem 3. But $T^k - A$ is a spanning subgraph of $G^k - A$; thus $G^k - A$ is Hamiltonian connected.

Corollary 3A proves Conjecture 1 for the case of $\kappa(G) = 1$. We can improve this result slightly in the direction of Conjecture 1 as follows:

COROLLARY 3B: *Let* G *be a connected finite graph and let* k *be an integer no less than 3. Let* $A \subseteq V(G)$ with $|A| \leq k + \kappa(G) - 4$. *Then* $G^k - A$ *is Hamiltonian connected.*

PROOF: A can be expressed as the union of disjoint sets B and C such that $|B| \leq k-3$ and $|C| \leq \kappa(G) - 1$. From the definition of $\kappa(G)$, $G - C$ is connected; thus $(G-C)^k - B$ is Hamiltonian connected by Corollary 3A. But $(G-C)^k$ is a spanning subgraph of $G^k - C$ and $B \cup C = A$; hence $(G-C)^k - B$ is a spanning subgraph of $G^k - A$, and $G^k - A$ is Hamiltonian connected.

LEMMA 4: *Let* k *be an integer no less than 2, let* G *be a graph, and let* $A \subseteq V(G)$ *such that* $|A| \leq k-2$. *Let* p *be an* (A, k)*-solid path in* G. *Then* $A \subseteq I(p)$.

PROOF: By definition, $l(p) = |I(p)| + 1$ and $|I(p) - A| \leq 2$. Therefore, $k < l(p) = |I(p)| + 1 = |I(p) \cap A| + |I(p) - A| + 1 \leq |I(p) \cap A| + 3$, or $k - 2 \leq |I(p) \cap A|$. But $|A| \leq k-2$. Thus, $A \subseteq I(p)$.

In a paper published in 1960, Sekanina [18] proved

THEOREM (Sekanina's Theorem): *If* G *is a connected finite graph, then* G^3 *is Hamiltonian connected.*

This theorem was proved again by Karaganis in a paper [15] published in 1968. Using this result, Chartrand and Kapoor [2] proved

THEOREM A: *If* G *is a connected finite graph with at least 4 vertices, then* G^3 *is 1-Hamiltonian.*

A proof by construction for the next theorem was recently published in [1]. However, the following proof is belived to have points of sufficient interest to warrant its publication. Recall that if p is a path, then $/p/$ is the graph whose edges and vertices are precisely those of p.

THEOREM 4: *Let* k *be an integer no less than 3, and let* G *be a connected finite graph with at least* k+1 *vertices. Then* G^k *is* r*-Hamiltonian for every integer* r *in* $\{0, 1, \ldots, k-2\}$.

PROOF: Let $A \subseteq V(G)$ with $|A| \leq k-2$. If $|A| \leq 1$, then $G^3 - A$ contains a Hamiltonian circuit by Theorem A. Since $G^3 - A$ is a spanning subgraph of $G^k - A$, $G^k - A$ is Hamiltonian.

Suppose $|A| \geq 2$. Since G is connected, we may choose a spanning tree T of G. If T has no (A, k)-solid paths, then $T^k - A$ is Hamiltonian connected by Theorem 3, and so it contains a Hamiltonian circuit. Since $T^k - A$ is a spanning subgraph of $G^k - A$, $G^k - A$ is Hamiltonian.

Now we assume that T contains an (A, k)-solid path a; by Lemma 4, $A \subseteq I(a)$. Since $|A| \geq 2$, there is an edge $\lambda = (\xi_1, \xi_2)$ in $E(a)$ such that each component of $/a/ - \lambda$ contains one or more vertices of A. Let T_i be the component of $T - \lambda$ which contains ξ_i, $i \in \{1, 2\}$. For $i \in \{1, 2\}$, if $\xi_i \notin A$, we let $\rho_i = \xi_i$, and if $\xi_i \in A$, we let ρ_i be a vertex of T_i which is not in A and which is A-joined to ξ_i (such a vertex exists because the ends of a are not in A). Let δ_i be a vertex of $V(T_i) - A$ different from ρ_i which is A-joined to ρ_i (if no such vertex exists, let $\delta_i = \rho_i$). Since both of T_1 and T_2 contain vertices of A, $|V(T_i) \cap A| < k-2$ for each i; hence T_i satisfies the conditions of Lemma 1 and contains no (A, k)-solid arc. Then applying Lemma 3 to T_i, $i \in \{1, 2\}$, let a_i be a Hamiltonian path in $T_i^k - A$ from ρ_i to δ_i (if $\rho_i = \delta_i$, let $a_i = \rho_i$). Since ρ_1 is the only vertex of $V(T) - A$ which can lie on the $[\delta_1, \rho_2]$-path of T, $d_T(\delta_1, \rho_2) \leq k$ by Lemma 2. Similarly, $d_T(\delta_2, \rho_1) \leq k$. Hence $(a_1), (a_2), \rho_1$ is a Hamiltonian circuit in $T^k - A$, and it is also a Hamiltonian circuit in $G^k - A$.

The following example shows that Theorem 4 cannot be strengthened to the level of Theorem 3, and Corollary 3A is best possible in the usual sense. In the tree T shown in figure 2A, the five vertices surrounded by the dashed curve are the vertices in A, and $k = 7$. $T^7 - A$ is shown in figure 2B. In T, α and β are A-joined, but it is easily shown that in $T^7 - A$, there is no Hamiltonian $[\alpha, \beta]$-path (γ would have to be both the successor of α and the predecessor of β in any such path).

The following corollary provides a slight weakening of the conditions of Theorem 4, with a corresponding weakening of the conclusion:

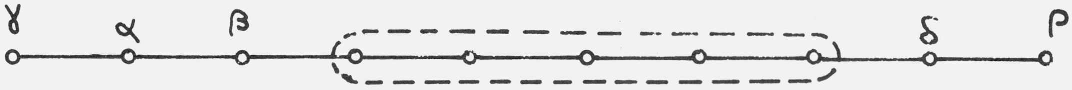


FIGURE 2A

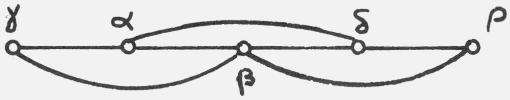


FIGURE 2B

COROLLARY 4A: *Let* G *be a connected finite graph with at least 3 vertices, let* k *be an integer no less than 3, and let* A *be a subset of* V(G) *such that* $|A| \leq min\ (k-1, |V(G)|-2)$. *Then* $G^k - A$ *contains a Hamiltonian path.*

PROOF: We simply remove all but one of the vertices of A from G^k. By Theorem 4, the resulting graph contains a Hamiltonian circuit. Removing one more vertex converts the circuit into a Hamiltonian path in the resulting graph, which is $G^k - A$.

While we have not proved Conjecture 2, Theorem 4 allows us to come a little closer to the statement of the conjecture, as the following corollary shows:

COROLLARY 4B: *Let* k *be an integer no less than 3 and let* G *be a connected finite graph with at least* $k + \kappa(G)$ *vertices. Then* G^k *is* r*-Hamiltonian for every* r *in* $\{0, 1, \ldots, k + \kappa(G) - 3\}$.

The proof of this corollary is similar to the proof of Corollary 3B.

Conjecture 2 might be extended to include the following:

CONJECTURE 3: *Let* G *be a graph with* $\kappa(G) \geq 2$ *and with at least* $2\kappa(G) + 1$ *vertices. Then* G^2 *is* $(2\kappa(G) - 2)$*-Hamiltonian.*

I suspect that Conjecture 3 does not hold (although it is valid if G is a circuitoid). However, the following weaker theorem was recently proved [3, 14]:

THEOREM: *Let* G *be a graph with* $\kappa(G) \geq 2$ *and with at least 4 vertices. Then* G^2 *is 1-Hamiltonian.*

4. Squares of Graphs

The set of all vertices of G having degree k is denoted by $V_k(G)$. A *caterpillar* is a tree T such that $T - V_1(T)$ is a pathoid or the empty graph. A *2-factor* of a graph G is a subgraph H of G such that $V(H) = V(G)$ and every vertex in H has degree 2.

In contrast to the situation for higher powers of graphs, we do not yet know which graphs have Hamiltonian squares. Neuman [16] has characterized those trees for which a Hamiltonian path joining two specified vertices will exist in the square of the trees. An easy consequence of his characterization is the following result:

THEOREM 5: *The square of a tree is Hamiltonian if and only if the tree is a caterpillar with at least 3 vertices.*[2]

H. Fleischner [6, 7, 8, 9] has proved that the square of a block is Hamiltonian, and he has characterized those cubic graphs with Hamiltonian squares [5, 10]. A few other more special results are known. The remainder of this paper contains two further theorems connected with the problem of determining which graphs have Hamiltonian squares.

[2] Neuman's result can also easily be used to show that the square of a tree is Hamiltonian connected if and only if the tree is a caterpillar with at most one vertex of degree greater than one.

If the square of a graph contains a Hamiltonian circuit, it certainly contains a 2-factor. Thus, it is reasonable to consider the question of which squares of graphs contain 2-factors ($S(K_{1,3})$ is an example of a graph whose square does *not* contain a 2-factor). In this connection, the following theorem is of interest. Given a path p, the *internal vertices* of p are the vertices in $I(p)$. We denote the degree in G of a vertex ξ by $v_G(\xi)$.

THEOREM 6: *Let* G *be a graph with minimum degree at least* 2. *Then* G^2 *contains a* 2*-factor.*

PROOF: By Theorem 5, it is sufficient to find a spanning forest in G in which each tree is a caterpillar with at least 3 vertices. We note first that since G has minimum degree at least 2, each component of G contains at least 3 vertices, and a longest path in G must have length at least 2.

Choose a longest path a_1 in G. Having chosen paths $a_1, \ldots, a_k$ in G, if $H_k = G - \bigcup_{i=1}^{k} V(a_i)$ is not empty, choose a longest path a_{k+1} in H_k. Note that the end vertices of a_{k+1} can be adjacent only to vertices in $\bigcup_{i=1}^{k} I(a_i) \cup V(a_{k+1})$. Eventually, since G is finite, we find we have chosen paths $a_1, \ldots, a_r$ such that H_r is the empty graph. Now for each end vertex μ of each path of length 0 or 1, choose an internal vertex of a longer path to which μ is adjacent through an edge λ_μ (such an internal vertex must exist since $v_G(\xi) \geqslant 2$ for all $\xi \in V(G)$). We form a spanning forest whose trees are caterpillars by deleting the edges from those pathoids in $\{/a_1/, \ldots, /a_r/\}$ which have length 1 and adding the edges λ_μ to form the caterpillars. Since each of these caterpillars includes the vertices of a path with an internal vertex, each caterpillar has at least three vertices.

An *Euler* graph is a graph in which every vertex has even degree. In a path $p = (r), \xi, (s)$, the *neighbors* of the vertex ξ are the vertices $L(r)$ and $F(s)$. Given a path $p = \xi_1, \xi_2, \ldots, \xi_{r-1}, \xi_r$, we let p^{-1} denote the path $\xi_r, \xi_{r-1}, \ldots, \xi_2, \xi_1$. We call a path x a *section* of a path p iff there exist paths y and z such that $p = (y), (x), (z)$ (y or z may be empty). Recall that all walks in this paper are denoted by sequences of vertices.

The next theorem is a beginning for the study of the question of which Euler graphs have Hamiltonian squares. Fleischner's proof that the square of a block is Hamiltonian involves finding a Hamiltonian circuit in the square of an Euler graph (see especially [7]); thus, a thorough understanding of Euler graphs with Hamiltonian squares would be quite desirable. The following Theorem 7, whose statement is quite long, can be summarized as follows: "If G is an Euler graph with at least 3 vertices such that $G - V_2(G)$ is a forest, then G^2 is Hamiltonian. Further, G^2 contains a Hamiltonian circuit which includes many of the edges of G."

THEOREM 7: *Let* G *be a connected Euler graph with at least* 3 *vertices such that* $G - V_2(G)$ *is a forest with trees* $T_1, \ldots, T_r$, $r \geqslant 0$. *Let* e *be an Euler trail in* G. *In each tree* T_i, $i \in \{1, 2, \ldots, r\}$, *choose one vertex* η_i *and call it the root of* T_i. *Then* G^2 *contains a Hamiltonian circuit* h *such that:*

(1) h *is a subsequence of* e, *where both* h *and* e *are denoted by a sequence of vertices;*

(2) *the root* η_i *of each tree* T_i *is adjacent in* G *to its two neighbors in* h; *and*

(3) *for each path* s *in* G *such that* $V(s) \subseteq V_2(G)$, *either* s *or* s^{-1} *is a section of* h.

PROOF: The theorem is trivial if G is a circuitoid. Thus we may assume $r \geqslant 1$. Define a function Ψ, with domain $V(G) - (V_2(G) \cup \{\eta_1, \eta_2, \ldots, \eta_r\})$, as follows: for $\xi \neq \eta_i$ in $V(T_i)$, let $\Psi(\xi)$ be the second vertex on the unique $[\xi, \eta_i]$-path in T_i. In e, mark each vertex of $V_2(G)$, and mark one occurrence of each of $\eta_1, \ldots, \eta_r$. Also for each vertex $\xi \in V(G) - (V_2(G) \cup \{\eta_1, \ldots, \eta_r\})$, mark precisely that occurrence of ξ in e for which $\Psi(\xi)$ is a neighbor of ξ at that location in e. Form h from e by deleting all unmarked vertices in e. This choice of h satisfies condition (1) of the theorem.

Since we left in h one occurrence of each of the distinct vertices in e, and since $V(e) = V(G)$, $V(h) = V(G)$. Note that the unmarked vertices of e are all in the forest $G - V_2(G)$. Two vertices of $G - V_2(G)$ which are successive in e and are adjacent in that subgraph are in the same tree T_i. Hence one is the second vertex in the path joining the other to η_i, so that one of them is necessarily marked at that part of e. Thus no two successive vertices in e are both unmarked, and for every section ω, χ of h, there is at most one vertex between ω and χ in e at the corresponding location in e. Since e is a trail in G, $d_G(\omega, \chi) \leqslant 2$. Thus, h is a Hamiltonian circuit in G^2.

For each $i \in \{1, 2, \ldots, r\}$, since η_i is $\Psi(\xi)$ for every vertex ξ adjacent to η_i in T_i, and since no vertex of degree 2 in G is deleted in obtaining h from e, no neighbors of η_i in e are deleted when forming h. Thus, in the one occurrence of η_i in h, both neighbors of η_i in h are neighbors of η_i in e and hence are adjacent to η_i in G. Finally, for each path s in G such that $V(s) \subseteq V_2(G)$, either s or s^{-1} is a section of e. Since no vertices of degree 2 in G are deleted in forming h, s or s^{-1} remains a section of h. Thus h satisfies conditions (2) and (3) of the theorem.

For any $\xi \in V(G)$, if ξ and η_i are adjacent in G they must be neighbors at some occurrence of η_i in e. Then at that occurrence of η_i, η_i can be marked. Thus, for each root η_i, we may select one vertex adjacent to η_i in G which will be a neighbor of η_i in h.

Let G be a graph whose square is Hamiltonian, let h be a Hamiltonian circuit in G, and let λ be an edge of G in h. Let ξ be one of the ends of λ. Further, let H_1 and H_2 be disjoint caterpillars disjoint from G, and let η_i be a vertex of degree no more than 1 in $H_i - V_1(H_i)$, for $i \in \{1, 2\}$ (if H_i has only two vertices, let η_i be one of them). Form a graph M from H_1 and G by identifying η_1 and ξ. Then M^2 is Hamiltonian as indicated by figure 3, where the dashed line indicates the sequence inserted between the ends of λ in h, (in this and following figures, a wavy line is used to denote h). Variations in the length of $H_1 - V_1(H_1)$ (particularly variations to odd lengths) and changes in the number of vertices of degree 1 in H_1 can be dealt with in an obvious manner. Note that one edge of H_1 is included in the Hamiltonian circuit of M^2.

Now let H_1 and H_2 be as before, but suppose G^2 contains a Hamiltonian circuit h which includes two edges λ and μ of G, both incident with the same vertex ξ of G. In this case, let N be formed from H_1, H_2, and G by identifying η_1, η_2, and ξ. Then N^2 contains a Hamiltonian circuit as indicated by figure 4, where the dashed lines indicate the sequence which is to replace ξ in h. Variations in H_1 and H_2 are again easily dealt with. Note that an edge of each of H_1 and H_2 is in the Hamiltonian circuit of N^2.

Now let G and H be disjoint graphs such that G^2 and H^2 are both Hamiltonian. Let λ be an edge of G incident with a vertex ξ and in a Hamiltonian circuit g of G^2, and let μ be an edge of H incident with a vertex η and in a Hamiltonian circuit h of H^2. Form a graph P from G and H by identifying ξ and η. Then P^2 contains a Hamiltonian circuit as indicated by the wavy and dashed curves in figure 5.

Theorem 7 describes many edges of G which are included in a Hamiltonian circuit of G^2, where G is an Euler graph in which $G - V_2(G)$ is a forest. In view of the techniques described in the preceding three paragraphs, Theorem 7 can thus be used to show many other graphs have Hamiltonian squares.

An Euler graph G whose square is not Hamiltonian is shown in figure 6. It is easy to see that any Hamiltonian circuitoid in G^2 must contain edges joining a vertex other than a cut vertex of G of each of the small triangles T_1, T_2, T_3, T_4, and T_5 to one of α or β. Thus one of α or β must meet three edges

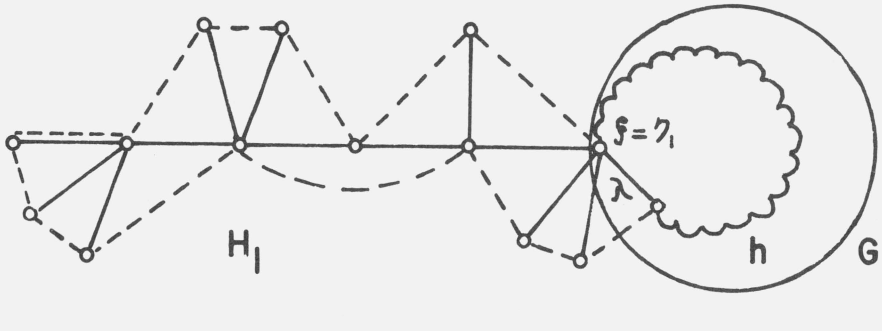


FIGURE 3

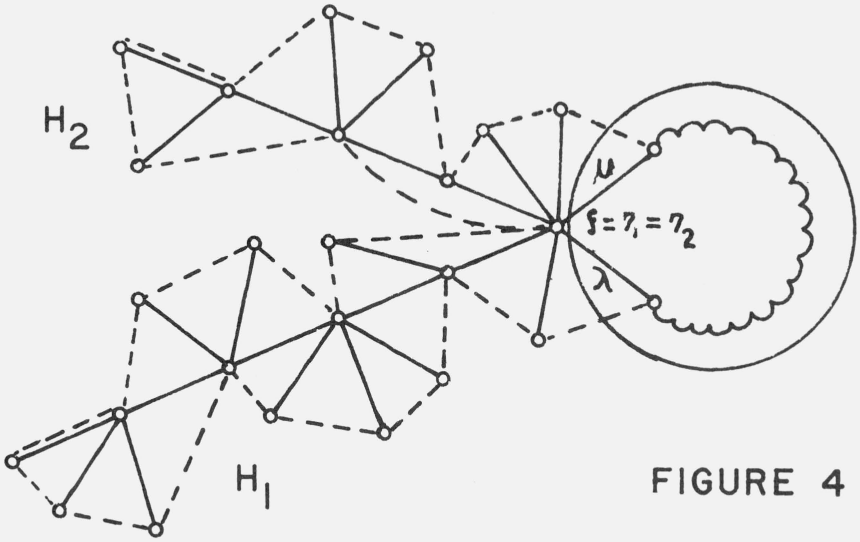


FIGURE 4

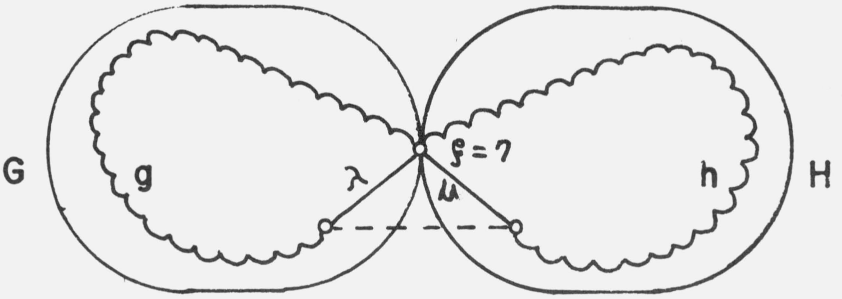


FIGURE 5

of the Hamiltonian circuitoid, which is impossible. I suspect that the graph of figure 6 is the smallest Euler graph whose square is not Hamiltonian (size being measured in number of vertices or number of edges).

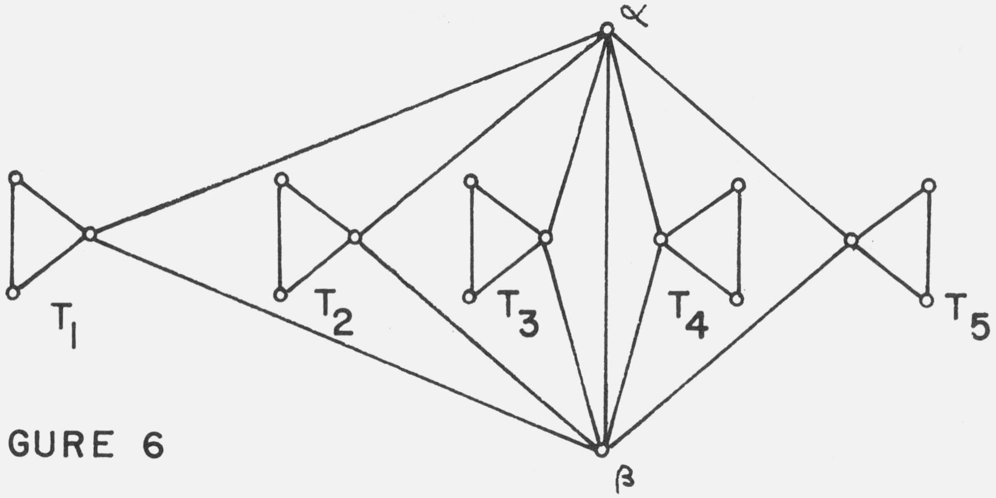


FIGURE 6

ADDED IN PROOF: Since this paper was submitted, references [6] and [7] have been combined into Fleischner, H., On spanning subgraphs of a connected bridgeless graph and their application to *DT*-graphs, J. Combinatorial Theory (to appear), and references [8] and [9] were combined into Fleischner, H., The square of every two-connected graph is Hamiltonian, J. Combinatorial Theory (to appear). Further, references [3] and [14] were combined into Chartrand, G., Hobbs, A. M., Jung, H. A., Kapoor, S. F., and Nash-Williams, C. St. J. A., The square of a block is Hamiltonian connected, J. Combinatorial Theory (to appear). It should be mentioned that the term "ξ-section" is similar to the term "J-component" of Tutte, W. T., The Connectivity of Graphs (Toronto Univ. Press, Toronto, 1967), and that many of the other terms in this paper not found in [11] were derived from the definitions in Nash-Williams, C. St. J. A., Graph-Theoretic Definitions (unpublished).

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5. References

[1] Bhat, V. N., and Kapoor, S. F., The powers of a connected graph are highly Hamiltonian, J. Res. Nat. Bur. Stand. (U.S.), **75B** (Math. Sci.) Nos. 1 & 2, 63–66 (Jan.–June 1971).
[2] Chartrand, G., and Kapoor, S. F., The cube of every connected graph is 1-Hamiltonian, J. Res. Nat. Bur. Stand. (U.S.), **73B** (Math. Sci.) No. 1, 47–48 (Jan.–Mar. 1969).
[3] Chartrand, G., and Kapoor, S. F., The square of every 2-connected graph is 1-Hamiltonian (to appear).
[4] Chartrand, G., Kapoor, S. F., and Lick, D. R., n-Hamiltonian graphs, J. Combinatorial Theory **9**, 308–312 (1970).
[5] Fleischner H., Über Hamiltonsche Linien im Quadrat kubishcer und pseudokubischer Graphen (to appear).
[6] Fleischner, H., On a certain spanning subgraph of a connected bridgeless graph (to appear).
[7] Fleischner, H., The total graph of a block is Hamiltonian (to appear).
[8] Fleischner, H., On line-critical blocks (to appear).
[9] Fleischner, H., The square of every non-separable graph is Hamiltonian (to appear).
[10] Fleischner, H., Zum Quadrat kubischer Graphen (to appear).
[11] Harary, F., Graph Theory (Addison-Wesley Publishing Co., Reading, Mass., 1969).
[12] Hobbs, A. M., Some Hamiltonian results in the square of a graph, Notices Amer. Math. Soc. **18,** 397 (1971).
[13] Hobbs, A. M., A catalog of minimal blocks (to appear).
[14] Hobbs, A. M., and Nash-Williams, C. St. J. A., The square of a block is Hamiltonian connected (to appear).
[15] Karaganis, J. J., On the cube of a graph, Canad. Math. Bull. **11,** 295–296 (1968).
[16] Neuman, F., On a certain ordering of the vertices of a tree, Časopis Pěst. Mat. **89,** 323–339 (1964).
[17] Ore, O., Hamiltonian connected graphs, J. Math. Pures Appl. **42,** 21–27 (1963).
[18] Sekanina, M., On an ordering of the set of vertices of a connected graph, Publ. Fac. Sci. Univ. Brno, Tchecoslovaquie, No. 412, 137–141 (1960).

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