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On a Functional Equation*

A. O. L. Atkin**

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Subject to suitable conditions of continuity and normalization, it is shown that the equation $f(x, y)+f(x+y, z)=f(y, z)+f(x, y+z)$ implies $f(x, y)=g(x)+g(y)-g(x+y)$. The result has application in physics to the motion of an electron in a crystal lattice.

Key words: Analysis; continuous; equation; function; real.

The object of this note is to prove Lemma 1. The motivation is given in Lomont and Moses;[1] I am grateful to Professor Lomont for bringing this problem to my notice.

LEMMA 1: *Suppose that* $f: R \times R \to R$ *is continuous and satisfies*

$$f(x, y)+f(x+y, z)=f(y, z)+f(x, y+z) \tag{1}$$

for all x, y, z, *in* R. *Then there exists an unique continuous function* $g: R \to R$ *such that* $g(0)=g(1)$ *and* $f(x, y)=g(x)+g(y)-g(x+y)$ *for all* x, y, *in* R.

Without loss of generality we may assume $f(0, 0)=0$ (otherwise $f'=f-k$, $g'=g-k$, reduces). We use (a, b, c) to denote the application of (1) with $x=a$, $y=b$, $z=c$. Then

$$\begin{aligned} &(x, 0, 0) \Rightarrow 2f(x, 0)=f(x, 0) \Rightarrow f(x, 0)=0 && \text{for all } x \text{ in } R. \\ &\text{Similarly} \qquad\qquad\qquad\qquad f(0, y)=0 && \text{for all } y \text{ in } R. \end{aligned} \tag{2}$$

We now prove

LEMMA 2: $f(x, y)=f(y, x)$ *for all* x, y, *in* R.

Define $F(x, y)=f(x, y)-f(y, x)$. Then $(x, y, x) \Rightarrow$

$$F(x, y)=F(x, x+y). \tag{3}$$

For nonnegative N in Z let $P(N)$ be the proposition:

"$F(mx, nx)=0$ for all x in R and all m, n, in Z with $|m| \leqslant N$, $|n| \leqslant N$."

Then P(0). Assume $P(k-1)$ for a given $k \geqslant 1$. Then for $|n| \leqslant k-1$ we have by (3) that

$$F(kx, nx)=-F(nx, kx)=-F(nx, (k-n)x),$$

and the last term is zero either by hypothesis $(n \neq 0)$ or by (2); replacing x by $-x$ gives $F(-kx, nx)=F(nx, -kx)=0$. Finally $F(kx, kx)=F(kx, 0)=F(kx, -kx)$ by (3), and the middle term is zero by

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** Present address: The University of Illinois at Chicago Circle, Chicago, Illinois 60680

[1] J. S. Lomont and Harry E. Moses, The Application of Ray Representations of Translation Groups to the Motion of an Electron in a Crystal Lattice, Annals of Physics 67(2), 1971, pp. 406–431.

(2); replacing x by $-x$ completes the proof of $P(k)$. Hence $P(N)$ for all N by induction.

Thus $F(x, y)=0$ whenever $x:y$ is rational, and in particular when x and y are both rational. So F is continuous on $R\times R$ and zero on a subset dense in $R\times R$, and hence identically zero. This proves Lemma 2.

The construction of g(x).

Denote by S_N the set of all rational points $a/2^N$, with a in Z and N nonnegative in Z. Then

$$S=\sum_{N=0}^{\infty} S_N \ . \ . \ . \supset S_N \supset \ . \ . \ . \supset S_1 \supset S_0.$$

We define a function $g:S\to R$ inductively as follows.

$$g(0)=g(1)=0;\ g(n)=g(n-1)-f(1, n-1)\,(n\geqslant 2);$$

$$g(n)=g(n+1)+f(1, n)\,(n\leqslant -1): \tag{4}$$

and for $N\geqslant 1$ and x in S_N-S_{N-1},

$$g(2^{-N})=g(2^{1-N})/2+f(2^{-N}, 2^{-N})/2;$$

$$g(x)=g(x-2^{-N})+g(2^{-N})-f(2^{-N}, x-2^{-N}). \tag{5}$$

LEMMA 3: *We have*

$$\mathrm{f(x, y)=g(x)+g(y)-g(x+y)}\ \textit{for all}\ \mathrm{x, y,}\ \textit{in}\ \mathrm{S}.$$

Define $F(x, y)=f(x, y)-g(x)-g(y)+g(x+y)$, so that $F(x, y)=F(y, x)$. Then by (4) $F(1, n)=0$ for all n in Z, and $F(2^{-N}, x)=0$ for all x in S_{N-1} and for $x=2^{-N}$, if $N\geqslant 1$, by (5). Also F clearly satisfies the functional equation (1). So for x, y, in $S_0(=Z)$ we have $(x, y, 1)\Rightarrow F(x, y)=F(x, y+1)$, whence $F(x, y)=0$ for all x, y in S_0.

Assume now that $F(x, y)=0$ for all x, y in S_{k-1}, for a given $k\geqslant 1$. Then if x is in S_k-S_{k-1}, we have $z=x-2^{-k}$ in S_{k-1}, and $(2^{-k}, 2^{-k}, z)\Rightarrow F(2^{-k}, x)=0$. Using this, for y, z, in S_k we have $(2^{-k}, y, z)\Rightarrow F(2^{-k}+y, z)=F(y, z)$, and these are now both equal to $F(2^{-k}+y, 2^{-k}+z)=F(y, 2^{-k}+z)$. One of the four equal terms has both arguments in S_{k-1}, and is zero by hypothesis. Thus we have shown that $F(y, z)=0$ for all y, z, in S_k. Lemma 3 now follows by induction.

LEMMA 4: Given $A>0$, $\epsilon>0$, there exists $\delta>0$ such that for all x, y, in S with $|x-y|<\delta$ and $|x|\leqslant A$, $|y|\leqslant A$, we have $|g(x)-g(y)|<\epsilon$.

First x in $S\Rightarrow 2x$ in S. Let $L=\limsup|g(x)|$ as $x\to 0$ in S. Then $g(2x)=2g(x)-f(x, x)$, so that $2L\leqslant L+\limsup_{x\to 0}|f(x, x)|=L$, and hence $L=0$ and $g(x)\to 0$ as $x\to 0$ in S. Now choose $\delta_1>0$ such that $|g(z)|<\epsilon/2$ for $|z|<\delta_1$, and $\delta_2>0$ such that $|f(x, z)|<\epsilon/2$ for $|z|<\delta_2$ and all x satisfying $|x|\leqslant A$, $|z+x|\leqslant A$; the latter choice is possible by the uniform continuity of f in a bounded region $R\times R$. Then with x, $z=y-x$, for x, y, in Lemma 3 we have for $|x-y|<\delta=\min(\delta_i, \delta_2)$

$$|g(x)-g(y)|<|f(x, z)|+|g(z)|<\epsilon,$$

as desired.

Extension of g(x) *to* R.

Given any x in R, choose x_n in S such that $x_n\to x$ as $n\to\infty$, possible since S is dense in R. As in the definition of real numbers using Cauchy sequences, Lemma 4 shows that $\lim_{n\to\infty} g(x_n)$ exists, and is independent of the choice of x_n, and we define $g(x)$ as this limit: $g(x)$ is also clearly continuous.

Finally we write $F(x,y)=f(x,y)-g(x)-g(y)+g(x+y)$ for x,y, in R. F is continuous on $R\times R$, and by Lemma 3 F is zero on $S\times S$ dense in $R\times R$; hence F is identically zero. This proves the existence in Lemma 1.

The uniqueness of $g(x)$ is equivalent to the proposition:

$$\textit{If } g{:}R\to R \textit{ is continuous}, \; g(0)=g(1)=0,$$

and $g(x+y)=g(x)+g(y)$ *for all* x,y, *in* R, *then* $g(x)\equiv 0$.

This follows by a routine repetition of the technique above, since starting with $x=0$ and $x=1$, the operations $x\to x/2$ and $x,y\to x-y$ generate S, a subset dense in R, while $g(x/2)=g(x)/2$ and $g(x-y)=g(x)-g(y)$.

Finally if $g(0)=g(1)$ is not specified in Lemma 1, we may add αx to $g(x)$, for any real constant α.

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