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Theory of Disclinations: II. Continuous and Discrete Disclinations in Anisotropic Elasticity

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A general theory of stationary disclinations for a linearly elastic, infinitely extended, homogeneous body is developed. Dislocation theory is extended in three different ways to include disclinations, i.e., from continuous distributions, discrete lines, and continuous distributions of infinitesimal loops. This leads to three independent definitions of the disclination, which can be uniquely related to each other. These interrelationships clarify Anthony and Mura's approaches to disclination theory, which at first appear to diverge from the present theory. Mura's "plastic distortion" and "plastic rotation" are identified as the dislocation and disclination loop densities. The elastic strain and bend-twist are derived as closed integrals in terms of the defect densities, and shown to be state quantities. The theory reduces to classical dislocation theory when the disclinations vanish. For every discrete disclination line, it is always possible to find a "dislocation model," which is a dislocation wall terminating on the line that gives exactly the same elastic strain and stress.

Key words: Burgers vector; continuum mechanics; defect; dipole; disclination; dislocation; distortion; Green's tensor; incompatibility; loop; plasticity; strain; Volterra.

Contents

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1. Introduction

This paper develops a general theory of stationary disclinations in a linearly elastic, infinitely extended, homogeneous body.

Nabarro [1][1] and Anthony [2] have already reviewed many of the geometrical aspects of discrete disclinations. The author [3] and Anthony [2] have also treated the geometry of continuous disclinations (i.e., continuous distributions of disclinations). We have a slight difference in interpretation with Anthony over the meaning of the dislocation density tensor, and the continuity equation for dislocations, in the presence of disclinations. This difference arises due to the essentially independent definitions of continuous and discrete defects, and we believe this problem has been resolved in the present paper. Furthermore, Anthony [2] has treated the statics of a wedge disclination in isotropic couple-stress theory, and also given a nonlinear formulation of the geometry of continuous disclinations.

The development of the theory we present owes much to Mura [4]. He has developed an anisotropic theory of discrete disclinations and generalized this to continuous distributions. He does not try to distinguish between geometry and statics. Mura introduced the concepts of "plastic distortion," $\boldsymbol{\beta}^*$, and "plastic rotation," $\boldsymbol{\phi}^*$, which we [3] claimed could not be defined when disclinations are present. However, these concepts can be incorporated into the general theory with a slight modification in interpretation: They can logically be regarded as the dislocation and disclination loop density tensors. Then these quantities, which have a clear definition for discrete defects, retain their physical significance when the theory is generalized to continuous distributions. In this sense the dislocation loop theory of Kroupa [5] is then extended to disclinations.

Mura also introduces an "elastic distortion" in the presence of disclinations. The physical significance of this quantity still has to be assessed, because it turns out not to be a *state quantity*, contrary to all previously defined elastic fields. By definition a state quantity is a continuously varying quantity which can be measured (in principle) by macroscopic experiments without any knowledge of the former states of the body. While for dislocations the elastic distortion *is* a state quantity, we find that with disclinations the elastic strain and bend-twist are the relevant state quantities. All of Mura's results [4] have been incorporated in the present paper.

There are several ways in which we can divide the theory of disclinations for the purpose of presentation. To clarify the logical structure of this paper we discuss them next, followed by an outline of the paper.

1.1. Dislocations Versus Disclinations

We can juxtapose the two types of defects. We first discuss the known results for dislocations to emphasize the new results that follow when disclinations are introduced. In this sense what has been called "theory of disclinations" in the literature is really a combined theory of disclinations and dislocations. For this reason, we shall use the term *defects* in this paper to denote the combi-

[1] Figures in brackets indicate the literature references at the end of this paper.

nation of both disclinations and dislocations.[2] In this terminology, then, this paper deals with continuous and discrete "defects" in anisotropic elasticity.

"Defect theory" is an extension of dislocation theory and reduces to it when the disclinations vanish. Although it is possible to have a pure "dislocation theory," it is not possible to have a "disclination theory" without dislocations. Nevertheless we left the word "disclination" in the title of this paper to emphasize that what is new is due to the introduction of disclinations.

1.2 Continuous Defects Versus Discrete Defect Lines Versus Defect Loop Densities

There are three essentially different ways to define defects:

The geometrical theory of continuous (distributions of) defects can be formulated by examining the consequences of violating the classical compatibility conditions [3]. In view of this a body with continuous defects is also called an incompatible body, whereas a defect-free body is called compatible.

On the other hand, Weingarten's theorem [3, 6] is the point of departure for the theory of discrete defect lines. This theorem provides the two characteristic constants associated with the discrete defect: The general Burgers vector, which reduces to the classical Burgers vector for a discrete dislocation, and the characteristic rotation vector of the discrete disclination. Like the Burgers vector for dislocations, the characteristic rotation vector plays just as important a role for disclinations. We have therefore ventured to call it the *Frank vector*, in honor of F. C. Frank, who coined the word "disclination."[3]

A third way to formulate defect theory is in terms of (a continuous distribution of infinitesimal) defect loop densities. This could be regarded as the simplest approach, because in general the loop densities can be arbitrarily prescribed. Furthermore any given defect can be built up from some loop distribution.

In relating the three concepts in the above three paragraphs complications arise. For dislocations there is a straightforward correlation between the dislocation density, the Burgers vector, and the dislocation loop density. When disclinations are introduced there is an analogous correlation between the disclination density tensor, the Frank vector, and the disclination loop density. However, it is found that there is no unique correlation between the dislocations defined in the three formulations, but that they get mixed with the disclinations. This is basically the source of our differences with Anthony (continuous versus discrete) and Mura (lines versus loops).

1.3. Geometry Versus Statics

These are the stationary equivalents of kinematics versus dynamics, a distinction made in almost all other fields of science.

The relations described under *geometry* headings simply result from the properties of Euclidean space and are independent of the properties of the body. The distinction between plastic and elastic under this heading is therefore arbitrary, but it acquires physical significance under statics. In every case, the results given under geometry in this paper are valid for a linearly elastic, homogeneous body, *finite* or *infinite*.

Under the *statics* headings the elastic quantities from geometry are related to the properties

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[2] This usage would seem to ignore the possibility of point defects. However, they can be represented in this theory by discrete defect loop densities. This will be illustrated by an example elsewhere [24].

[3] A historical remark is appropriate here: After Weingarten [6] published his theorem, Volterra [7] recognized its implications for the discrete defects, which he called "distortions." This would then at first seem like the best word to use for disclinations and dislocations combined. Kröner [8] did refer to them as "Volterra distortions," but he subsequently used the term "distortion" for the gradient of the displacement, a usage which has now become widely accepted. Meanwhile Love [9] ventured to call them "dislocations." This name has stuck, but since the translational type defect played the more important role in plasticity, the name "dislocation" gradually became more and more associated with this type of defect. It then became necessary to distinguish between translational and rotational dislocations, or, referring back to Volterra, between dislocations of the first, second, and third order, and dislocations of the fourth, fifth, and sixth order. Therefore, Frank [10] coined the term "disinclinations" for the latter. He subsequently modified it to "disclinations." Now the names "dislocations" and "disclinations" are becoming well-established in the literature, but so far no suitable term has been introduced to describe the combination of both of them, which we have simply called "defects." For additional information see Nabarro [3], pp. 17–20. We also add a remark on the nomenclature: In the present linear theory the rotation is traditionally represented by a vector, and successive rotations commute, in accordance with the usual vector addition rules. When the theory is extended to the nonlinear range the rotations can be finite and may no longer commute. In this case the rotation can still be represented by a vector (sometimes called *versor*), which obeys a more general addition rule. See section 5.1.

of the body. We confine ourselves to linear elasticity, i.e., Hooke's law, and a homogeneous body. Furthermore the specific results in this paper are limited by boundary conditions. These are that the body is infinitely extended and that the displacement and strain fields vanish at infinity faster than r^{-1} and r^{-2}, respectively.

1.4. Outline of Paper

In section 2 we derive a general solution of the plastic strain problem, which can be posed without specifying the nature of the defects involved. This problem is a generalization of Eshelby's "transformation problem" [11]. The result forms the basis for all subsequent applications to statics. In the derivation the general boundary conditions are introduced. Also the useful Green's tensor is defined. We show how Eshelby's method of solution can be generalized. Finally we show that a compatible plastic strain gives no elastic fields.

In section 3 we review classical dislocation theory to set the stage for the following sections.

In section 4 we derive the fields for a continuous distribution of defects. The geometry for this case has already been treated [3] and equations quoted from this reference are denoted by (I1), (I2), etc. In this section the useful incompatibility source tensor of Simmons and Bullough [12] is used to find the elastic strain and show that it is a state quantity.

Section 5 treats the discrete defect line. Weingarten's theorem is used to motivate the definition. Then the appropriate plastic strain and bend-twist are found, which are logically expressed in terms of Mura's $\boldsymbol{\beta}^*$ and $\boldsymbol{\phi}^*$. The dislocation density tensor is found to depend on the Frank vector, an example of the mixing referred to above. The static results are derived.

Section 6 shows the formulation in terms of a continuous distribution of infinitesimal defect loops. Here we identify $\boldsymbol{\beta}^*$ and $\boldsymbol{\phi}^*$, introduced for the discrete defect, with the dislocation and disclination loop density tensors. Hence this gives meaning to them in Mura's generalization to continuous distributions. As another example of mixing we find that the dislocation density depends on the disclination loop density.

In section 7 we derive some results for the discrete dipole line, and show how they are related to the dislocation dipole.

Section 8 shows that the general results of sections 4–6 reduce to those of section 3 when no disclinations are present.

Section 9 examines more closely the meaning of $\boldsymbol{\beta}^*$ introduced for a discrete defect loop. Without $\boldsymbol{\phi}^*$, it is shown to represent a terminating dislocation wall, i.e., a constant dislocation density on a surface terminating at the discrete dislocation line. This has been called the "dislocation model" of the discrete defect line. The elastic fields are derived.

Section 10 examines the meaning of $\boldsymbol{\phi}^*$ for a discrete loop. Without $\boldsymbol{\beta}^*$, it is shown to represent a compensated disclination loop, i.e., a surface with a constant dislocation density (opposite to that of sec. 9) terminating at a discrete disclination loop. The elastic fields of this defect vanish.

In an appendix we develop a special notation to deal with delta functions on curves and surfaces. This notation is very convenient and simplifies the equations that occur when we treat discrete defects.

Throughout the development of this paper we find that many concepts or quantities from dislocation theory generalize into pairs in defect theory. For example, dislocations generalize to defects, consisting of disclinations and dislocations, or the Burgers vector generalizes to the characteristic vectors, consisting of the Frank vector and the total Burgers vector. We have found it useful to introduce the new concept of "basic fields," consisting of the strain and bend-twist. Then the distortion in dislocation theory generalizes to the basic fields in defect theory. The nomenclature that has developed is summarized in tables 1 and 2.

This paper basically addresses itself to solving boundary value problems. The important subject of the forces on and the energy of the defects introduces additional complications. It is therefore omitted and will be treated elsewhere.

Neither shall we treat applications to special problems in the present paper. These will also be left for future publications [25, 26].

2. General Solution of the Plastic Strain Problem

2.1. Statement of the Problem

In this section we give a formal solution of the following problem, which can be posed without specifying the nature of the defects involved: given an infinitely extended homogeneous anisotropic body with the plastic strain e^P_{kl} given as a prescribed function of space. To find the resulting total displacement u^T_m throughout the body.

This problem is a generalization of Eshelby's [11] "transformation problem" to an anisotropic medium and an inhomogeneous stress free strain. We remark here that the concept of "stress free strain" is identical to that of plastic strain.

The statement of this problem can be rephrased in a manner that is almost identical to that of the classical problem of elasticity, i.e., in terms of the equation of equilibrium, Hooke's law, and the definition of strain. Using this approach we shall derive the solution to both problems simultaneously.

The equation of equilibrium for the stress σ_{ij} is:[4]

$$\sigma_{ij,i}+f_j=0,\ (i,\ j=1,\ 2,\ 3), \tag{2.1}$$

where f_i is the body force per unit volume. Here we have used the Einstein summation convention, and the subscripted comma followed by the index i indicates partial differentiation: $\partial\sigma_{ij}/\partial x_i$. The stress is related to the elastic strain e_{kl} by Hooke's law:

$$\sigma_{ij}=C_{ijkl}e_{kl}, \tag{2.2}$$

where the C_{ijkl} are the anisotropic elastic constants. Since σ_{ij} and e_{kl} are symmetric it follows that [5]

$$C_{ijkl}=C_{jikl}=C_{ijlk}. \tag{2.3}$$

The total strain e^T_{kl} is defined by [6]

$$e^T_{kl}\equiv u^T_{(l,k)}. \tag{2.4}$$

In general the total deformation is not completely elastic, but part of it is stress free or plastic, so that

$$e^T_{kl}=e_{kl}+e^P_{kl}. \tag{2.5}$$

The above relations are conveniently combined into the expression

$$C_{ijkl}u^T_{l,ki}+f_j=C_{ijkl}e^P_{kl,i}.$$

This is the set of partial differential equations we wish to solve for u^T_l when f_j and e^P_{kl} are given.

2.2. Definition and Application of Green's Tensor

To integrate the eq (2.6) it is useful to introduce Green's tensor function $G_{jn}(\mathbf{r})$, which represents the displacement in the x_j direction at the field point $\mathbf{r}$ arising from a point force in the x_n direction at the origin. Thus G_{jn} for an infinitely extended body is defined by

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[4] Cartesian coordinates are used for simplicity.

[5] If an elastic strain energy function exists, we also have the symmetry condition $C_{ijkl}=C_{klij}$, but we do not need this relation in the present paper.

[6] We define the symmetric part of a tensor T_{ij} by $T_{(ij)}=1/2\ (T_{ij}+T_{ji})$. When T_{ij} is a complicated expression involving many other subscripts we shall also write equivalently $(T_{ij})_{(ij)}$.

$$C_{ijkl}G_{jn,ik}(\mathbf{r})+\delta_{ln}\delta(\mathbf{r})=0 \tag{2.7}$$

together with boundary condition that G_{jn} vanish at infinity.[7] Here δ_{ln} is the Kronecker delta and $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function. The latter is defined in appendix B, where we also show that it is homogeneous of degree (-3) in r. Therefore it follows from (2.7) and the boundary conditions that $G_{jn}(\mathbf{r})$ is homogeneous of degree (-1), i.e., $G_{jn}(\mathbf{r})$ varies as r^{-1}.

We can now derive the solution of (2.6) in terms of Green's tensor. Writing for the relative radius vector

$$\mathbf{R}=\mathbf{r}-\mathbf{r}' \tag{2.8}$$

we have by (B3) and (2.7)

$$\begin{aligned} u_n^T(\mathbf{r}) &= \int \delta_{ln}\delta(\mathbf{R})u_l^T(\mathbf{r}')dV' \\ &= -\int C_{ijkl}G_{jn,ik}(\mathbf{R})u_l^T(\mathbf{r}')dV', \end{aligned} \tag{2.9}$$

where the integrations are taken over all space. Now, for any tensor $T(\mathbf{R})$, which is a function of $\mathbf{R}$ only, it follows from (2.8) that

$$T_{,i}(\mathbf{R})=-T_{,i'}(\mathbf{R})\equiv -\partial T/\partial x_i'. \tag{2.10}$$

Therefore we can also write (2.9) as follows[8]

$$u_n^T(\mathbf{r})=-\int C_{ijkl}G_{jn,i'k'}(\mathbf{R})u_l^T(\mathbf{r}')dV'. \tag{2.11}$$

By the divergence theorem, (A1) in appendix A, this relation can be transformed into

$$u_n^T(\mathbf{r})=-\oint C_{ijkl}G_{jn,i'}(\mathbf{R})u_l^T(\mathbf{r}')dS_k'+\oint C_{ijkl}G_{jn}(\mathbf{R})u_{l,k'}^T(\mathbf{r}')dS_i'-\int C_{ijkl}G_{jn}(\mathbf{R})u_{l,k'i'}^T(\mathbf{r}')dV', \tag{2.12}$$

where the surface integrals are taken at infinity. This step is usually called partial integration. The above relations hold only if the integrals converge.

We now assume the following boundary condition: The total displacement $u_l^T(\mathbf{r})\rightarrow 0$ as $r\rightarrow\infty$. Then $u_{l,k}^T(\mathbf{r})$ will approach zero faster than r^{-1} as $r\rightarrow\infty$. Hence in view of the behavior of G_{jn}, the integrands of the surface integrals in (2.12) will approach zero faster than $(r')^{-2}$ as $r'\rightarrow\infty$, and so these integrals will vanish. With this condition we also see that the volume integrals in (2.9), (2.11), and (2.12) converge. Thus we have by (2.6)

$$u_n^T(\mathbf{r})=\int G_{jn}(\mathbf{R})[f_j(\mathbf{r}')-C_{ijkl}e_{kl,i'}^P(\mathbf{r}')]dV'. \tag{2.13}$$

2.3. Solution of Classical Elasticity Problem

In classical elasticity, which we also call compatible theory, there is no plasticity, $e_{kl}^P=0$, and therefore (2.13) reduces to

$$u_n^T(\mathbf{r})=\int G_{jn}(\mathbf{R})f_j(\mathbf{r}')dV'. \tag{2.14}$$

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[7] If $C_{ijkl}=C_{klij}$, we also have the symmetry relation $G_{jn}=G_{nj}$ for an infinitely extended body, but we do not need this relation in the present paper.

[8] In (2.11) the Einstein summation convention also applies between the primed and unprimed indices.

This is the well-known classical solution, which is almost obvious if we remember the meaning of Green's tensor. In view of (2.6) and the boundary condition on u_l^T, it is necessary to assume the following condition: The prescribed body force $f_j(\mathbf{r})$ must approach zero faster than r^{-2} as $r \to \infty$. This requirement also insures that the integrand of (2.14) approaches zero faster than $(r')^{-3}$ as $r' \to \infty$, so that the integral is finite.

2.4. Solution of the Plastic Strain Problem

In the present problem, which we also call incompatible theory, there is plasticity, but no body force, $f_j = 0$, and therefore (2.13) reduces to

$$
\begin{aligned}
u_n^T(\mathbf{r}) &= -\int C_{ijkl} G_{jn}(\mathbf{R}) e_{kl,i'}^P(\mathbf{r}') dV' \\
&= -\int C_{ijkl} G_{jn,i}(\mathbf{R}) e_{kl}^P(\mathbf{r}') dV'
\end{aligned} \tag{2.15}
$$

by another partial integration and (2.10). Again we have taken the surface integral to vanish in the partial integration, and for this it is sufficient to assume the following condition: The prescribed plastic strain $e_{kl}^P(\mathbf{r})$ must approach zero faster than r^{-1} as $r \to \infty$. This requirement is consistent with the boundary condition on u_l^T in view of (2.6). Furthermore, the integrand of (2.15) will then approach zero faster than $(r')^{-3}$ as $r' \to \infty$, so that the integral is finite.

Equation (2.15) applies to any defect which can be described by the given plastic strain. It forms the basis for all subsequent applications to statics, where the same boundary condition on μ_l^T and the condition on e_{kl}^P must be satisfied. A similar result was derived by Mura [13], for a time-dependent plastic deformation. The above derivation emphasizes that it is only necessary to know the plastic strain (and not the plastic distortion), and it gives the condition it has to satisfy, as well as the boundary condition on the displacement, to find the solution for an infinitely extended body.

In the subsequent developments the results under geometry will hold regardless of the behavior at infinity, but those under statics are subject to the boundary conditions stated here because of the partial integrations involved.

2.5. Eshelby's Method

Physical science abounds with so-called "tricks" used to overcome mathematical difficulties. During the development of the mathematical theory of dislocations, Eshelby's bag of tricks has been remarkable. We wish to show how one of his recipes can be generalized to obtain (2.15).

He calculates the displacement for his transformation problem with the help of a sequence of imaginary cutting, straining and welding operations [11]. He does this so that he can introduce a fictitious body force simultaneously with the transformed inclusion such that there is no displacement. Then, he removes the body force and finds the resulting displacement from the classical expression.

To generalize this approach to our case it is presumably necessary to conceive of a continuous distribution of cutting, straining and welding operations. It is not important whether or not this can be imagined, because the formal steps are the same as in Eshelby's recipe. With the plastic strain e_{kl}^P we introduce a fictitious body force given by

$$
f_j = C_{ijkl}\, e_{kl,i}^P. \tag{2.16}
$$

Then (2.13) shows that there is no displacement. Next, we remove this body force by applying an equal but opposite body force $-f_j = -C_{ijkl}\, e_{kl,i}^P$ to the body. From (2.14) this leads to the displacement

$$
u_n^T(\mathbf{r}) = -\int C_{ijkl}\, G_{jn}(\mathbf{R})\, e_{kl,i'}^P(\mathbf{r}')\, dV'
$$

in agreement with (2.15). So the moral of Eshelby's method is that a defect described by the plastic strain e^P_{kl} can be simulated by the fictitious body force given by $-C_{ijkl}\, e^P_{kl,i}$.

2.6 Compatible Plastic Strain

There is an interesting consequence of (2.15) if the plastic strain is derivable from a plastic displacement

$$e^P_{kl} = u^P_{(l,k)}. \tag{2.17}$$

This will be called a compatible plastic strain. Then we have

$$\begin{aligned} u^T_n(\mathbf{r}) &= -\int C_{ijkl}\, G_{jn,i}(\mathbf{R})\, u^P_{l,k'}(\mathbf{r}')\, dV' \\ &= -\int C_{ijkl}\, G_{jn,ik}(\mathbf{R})\, u^P_l(\mathbf{r}')\, dV' \\ &= \int \delta_{ln}\, \delta(\mathbf{R})\, u^P_l(\mathbf{r}')\, dV' \\ &= u^P_n(\mathbf{r}). \end{aligned} \tag{2.18}$$

Here the second equality follows by a partial integration, and the third by (2.7). It follows therefore that when the plastic strain is compatible, the elastic displacement vanishes:

$$u_n = u^T_n - u^P_n = 0. \tag{2.19}$$

Hence, in this case all elastic fields vanish.

3. Review of Dislocation Theory

3.1. Continuous Distribution of Dislocations

3.1.1. Geometry

If in addition to the plastic strain e^P_{kl}, the plastic rotation ω^P_q is also prescribed, then we can identify the defects as dislocations. In this case we can say that the plastic distortion (I5.1) [9]

$$\beta^P_{kl} = e^P_{kl} + \epsilon_{klq}\omega^P_q \tag{3.1}$$

is prescribed arbitrarily as a function of space, where ϵ_{klq} is the permutation symbol. Here e^P_{kl} is the symmetric part of β^P_{kl}

$$e^P_{kl} = \beta^P_{(kl)}, \tag{3.2}$$

and the second term in (3.1) is the antisymmetric part of β^P_{kl}, or

$$\omega^P_q = 1/2\, \epsilon_{klq}\beta^P_{kl}\ . \tag{3.3}$$

The dislocation density is defined by (I5.2)

$$\alpha_{pl} \equiv -\epsilon_{pmk}\, \beta^P_{kl,m}, \tag{3.4}$$

or, equivalently,

$$\epsilon_{pmk}\alpha_{pl} \equiv \beta^P_{ml,k} - \beta^P_{kl,m}. \tag{3.5}$$

[9] The symbol (I5.1) refers to eq (5.1) in reference [3].

This definition implies we are using sign convention A, or **FS**/RH [14]. The continuity condition for dislocations (I5.4)

$$\alpha_{pl,\,p} = 0 \tag{3.6}$$

follows directly from (3.4). It implies that dislocations can not end inside the body.

We define a Burgers circuit as any closed curve λ inside the body (fig. 1). The Burgers vector

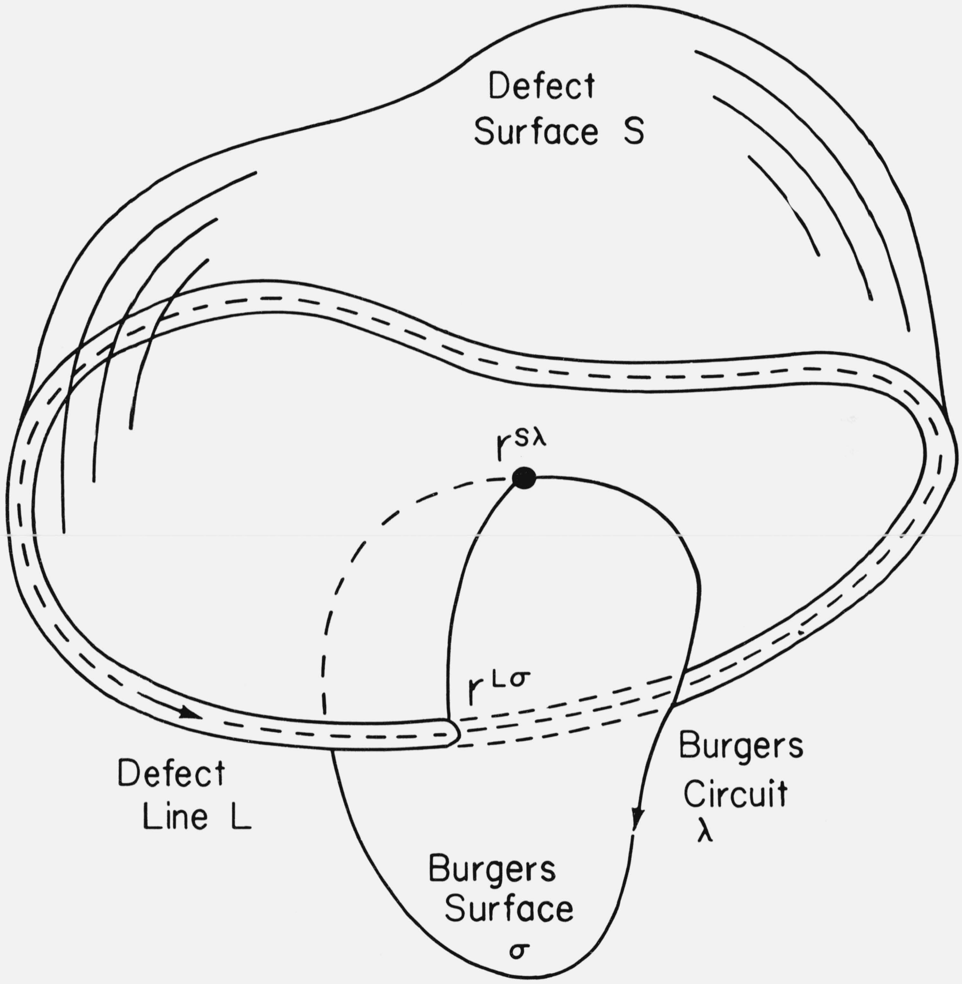

FIGURE 1. *Geometry of the discrete defect line* L, *its defect surface* S, *a Burgers circuit* λ, *and its Burgers surface* σ.

The surfaces S and σ are arbitrary and curved, subject only to the condition that they terminate on L and λ. The intersection of L with σ is labeled $\mathbf{r}^{L\sigma}$ and λ crosses S at $\mathbf{r}^{S\lambda}$. These are positive crossings by the right-hand rule.

associated with λ is defined by the closed line integral

$$b_l \equiv -\oint_\lambda \beta^P_{kl} dL_k . \qquad \text{(3.7}$$

This relation can be interpreted as follows: Starting with a perfect crystal we can imagine that the plastic deformation is produced by letting dislocations migrate into the crystal. A number of dislocations cut through λ, producing relative displacements $-\beta^P_{kl}dL_k$ in the lattice at the curve. By adding these contributions we measure the resultant Burgers vector of all the dislocations that remain stuck through the surface σ bounded by λ. From (3.4) and Stokes' theorem (A2) we can also write

$$b_l = \int_\sigma \alpha_{pl} dS_p. \tag{3.8}$$

This relation shows that the dislocation density α_{pl} represents the flux of dislocation (or Burgers vector) in the x_l direction that crosses unit area of a plane normal to the x_p direction.

The existence of the plastic distortion implies the existence of the elastic distortion (I2.1 and I5.5)

$$\beta^T_{mn} \equiv u^T_{n,\,m} = \beta_{mn} + \beta^P_{mn}. \tag{3.9}$$

Since the existence of plastic distortion implies there are no disclinations, this relation holds only if there are no disclinations present. This relation also allows us to draw the following conclusions from the definitions (3.4) and (3.7)

$$\epsilon_{pmk}\, \beta_{kl,\,m} = \alpha_{pl}, \tag{3.10}$$

$$\oint_\lambda \beta_{kl}\, dL_k = b_l. \tag{3.11}$$

These relations are called the basic geometric laws or field equations for α_{pl} and b_l.

Other quantities, which we shall find useful later, are the plastic bend-twist (I5.16)

$$\kappa^P_{kq} \equiv \omega^P_{q,\,k} = 1/2\, \epsilon_{lqr}\, \beta^P_{rl,\,k}, \tag{3.12}$$

where the last equality follows from (3.3), the elastic strain

$$e_{mn} = \beta_{(mn)}, \tag{3.13}$$

and the elastic bend-twist (I5.18)

$$\kappa_{st} \equiv \omega_{t,\,s} = 1/2\, \epsilon_{tmn}\, \beta_{mn,\,s}. \tag{3.14}$$

3.1.2. Statics

From (3.2) and (2.3) we find that the displacement (2.15) becomes

$$u^T_n(\mathbf{r}) = -\int C_{ijkl} G_{jn,\,i}(\mathbf{R})\, \beta^P_{kl}(\mathbf{r}')\, dV', \tag{3.15}$$

where $\beta^P_{kl}(\mathbf{r})$ must satisfy the condition that it approaches zero faster than r^{-1} as $r \to \infty$. From this relation we find the total distortion

$$\begin{aligned} u^T_{n,\,m}(\mathbf{r}) &= -\int C_{ijkl} G_{jn,\,im}(\mathbf{R})\, \beta^P_{kl}(\mathbf{r}')\, dV' \\ &= -\int C_{ijkl} G_{jn,\,i}(\mathbf{R})\, \beta^P_{kl,\,m'}(\mathbf{r}')\, dV' \\ &= \int C_{ijkl} G_{jn,\,i}(\mathbf{R})\, [\epsilon_{pmk} \alpha_{pl}(\mathbf{r}') - \beta^P_{ml,\,k'}(\mathbf{r}')]\, dV' \\ &= \int \epsilon_{pmk} C_{ijkl} G_{jn,\,i}(\mathbf{R})\, \alpha_{pl}(\mathbf{r}')\, dV' + \beta^P_{mn}(\mathbf{r}). \end{aligned} \tag{3.16}$$

Here the first equality follows simply by differentiating under the integral sign, where Green's tensor is the only function depending on $\mathbf{r}$, the second equality by a partial integration, the third from (3.5), and the fourth by a partial integration, (2.7) and (B3). From (3.9) we obtain the elastic distortion

$$\beta_{mn}(\mathbf{r}) = \int \epsilon_{pmk} C_{ijkl} G_{jn,\,i}(\mathbf{R})\, \alpha_{pl}(\mathbf{r}')\, dV'. \tag{3.17}$$

This is Mura's half of the Mura-Willis formula [12, 13, 15]. Willis' half would be included if (2.14) had also been included in the derivation. We note that from the above condition on β^P_{kl} and (3.4) that $\alpha_{pl}(\mathbf{r})$ must vanish faster than r^{-2} as $r \rightarrow \infty$. However, for the integral (3.17) to exist it is only necessary that $\alpha_{pl}(r)$ vanish faster than r^{-1} as $r \rightarrow \infty$.

Here we note that α_{pl} is a state quantity because it can be measured in the present state of the body, (e.g., by electron microscopy or x rays). Therefore (3.17) shows that β_{mn} is also a state quantity, because it is expressed entirely in terms of α_{pl}. On the other hand, for example, β^P_{mn} may not be a state quantity, because one may not be able to measure it without knowledge of the former states of the body.

3.2. The Discrete Dislocation Line

3.2.1. Geometry

The discrete dislocation line L is defined as the boundary of a surface S, where the material below S has been plastically displaced with respect to the material above S by an amount given by the constant Burgers vector b_l (fig. 1).

Hence, the difference between the displacement just below and above S is given by

$$[u_l(\mathbf{r})] = b_l. \tag{3.18}$$

Our problem now is to find the corresponding plastic distortion. The following is a straightforward procedure we have developed to find it. Assume first that S is closed, enclosing the volume V. Then by (B7)

$$u^T_l(\mathbf{r}) = \delta(V)\, b_l \tag{3.19}$$

represents a displacement that is equal to b_l inside V and has the jump (3.18) at the surface S. The corresponding distortion is by (3.9)

$$\begin{aligned} \beta^T_{kl}(\mathbf{r}) &= u^T_{l,\,k}(\mathbf{r}) \\ &= \delta_{,\,k}(V)\, b_l \\ &= -\delta_k(S)\, b_l, \end{aligned} \tag{3.20}$$

where we have used the divergence theorem (B24). Here S is the closed boundary of V. We see that the distortion is concentrated at the surface S. Since this deformation is just a rigid translation of part of the body, there is no elastic distortion, and therefore the above distortion is all plastic. We now simply generalize this expression to the open surface S of the dislocation loop:

$$\beta^P_{kl}(\mathbf{r}) = -\delta_k(S)\, b_l. \tag{3.21}$$

The Burgers vector for continuous distributions of dislocations was defined in (3.7). We want to show that this definition agrees with the constant b_l introduced above. If λ is any closed curve that encircles L in the positive sense (fig. 1), it will cross S positively at some point $\mathbf{r}^{S\lambda}$. Therefore

$$-\oint_\lambda \beta^P_{kl}\, dL_k = \oint_\lambda \delta_k(S)\, b_l dL_k = b_l \tag{3.22}$$

by (3.21) and (B15).

Now we find the dislocation density from (3.4) and Stokes' theorem (B26)

$$\begin{aligned}\alpha_{pl}(\mathbf{r}) &= \epsilon_{pmk}\, \delta_{k,\, m}(S)\, b_l \\ &= \delta_p(L)\, b_l,\end{aligned} \tag{3.23}$$

where L is the closed boundary of S, i.e., the dislocation line. The vector $\delta_p(L)$ is the Dirac delta function on the curve L and it is always parallel to L. A discrete dislocation line is called screw or edge when the Burgers vector is parallel or normal to the line, respectively. Therefore (3.23) shows that the diagonal and off-diagonal components of α_{pl} represent the screw and edge component of the dislocation density, respectively (see table 1). Equation (3.23) shows how to make the transition

TABLE 1. *Definition of various defect quantities*

Quantity	Dislocation / Disclination	Name of component: Diagonal	Name of component: Off-diagonal
Density tensor for continuous distribution of defects.	$\boldsymbol{\alpha}$ $\boldsymbol{\theta}$	screw wedge	edge twist
Density tensor for continuous distribution of infinitesimal defect loops.	$\boldsymbol{\gamma}$ or $\boldsymbol{\beta}^*$ $\boldsymbol{\zeta}$ or $\boldsymbol{\phi}^*$	prismatic twist	slip wedge

from a continuous distribution of dislocations to a discrete dislocation line. We note that it satisfies the continuity condition (3.6)

$$\alpha_{pl,\, p}(\mathbf{r}) = \delta_{p,\, p}(L)\, b_l = 0, \tag{3.24}$$

by (B28). As a cross-check we also show that (3.8) is satisfied by (3.23) (fig. 1)

$$\int_\sigma \alpha_{pl}\, dS_p = \int_\sigma \delta_p(L)\, b_l dS_p = b_l, \tag{3.25}$$

where we have used (B15) again. This last relation remains valid for many dislocation lines, and therefore can be used to show how to make the transition from discrete dislocation lines to a continuous distribution of dislocations: For many dislocation lines the average dislocation density α_{pl} represents the x_l component of the sum of the Burgers vectors of all the dislocation lines that intersect unit area of a plane normal to the x_p direction. For another interpretation of α_{pl} consider the result

$$\begin{aligned}\int_V \alpha_{pl}(\mathbf{r})\, dV &= \int \delta_p(L)\, b_l\, dV \\ &= b_l \int \delta_p(L)\, \delta(V)\, dV \\ &= b_l \oint_L \delta\,(V)\, dL_p \\ &= b_l \int_{L(V)} dL_p.\end{aligned} \tag{3.26}$$

Here the first equality follows from (3.23), the second from (B7), the third from (B11), and the fourth from (B7), where $L(V)$ is the part of the curve L inside V only. From this expression we see that the average dislocation density α_{pl} also represents the sum of the x_l component of the Burgers vectors times the projected line length in the x_p direction of all the dislocation lines per unit volume. The equivalence of the above two interpretations of the average dislocation density can also be shown by the methods of quantitative stereology [16].

We wish to point out here that there is also another type of dislocation density in wide use, primarily by experimentalists. This is the total dislocation line length per unit volume, usually designated by ρ:

$$\int_V \rho(\mathbf{r})\, dV = \int_{L(V)} dL. \tag{3.27}$$

It is easy to show that

$$\rho(\mathbf{r}) = t_p \delta_p(L), \tag{3.28}$$

where t_p is the unit tangent to the dislocation line. Hence the relation between the two different dislocation densities is

$$\rho b^2 = t_p \alpha_{pl} b_l \tag{3.29}$$

from (3.23).

3.2.2. Statics

Now we substitute (3.21) into (3.15) to find the displacement of a discrete dislocation line.

$$\begin{aligned} u_n^T(\mathbf{r}) &= \int C_{ijkl} G_{jn,\,i}(\mathbf{R})\, \delta_k(S')\, b_l dV' \\ &= \int_S C_{ijkl} G_{jn,\,i}(\mathbf{R})\, b_l dS'_k, \end{aligned} \tag{3.30}$$

using (B12). This equation allows us to estimate the asymptotic behavior of the displacement at large distances from a small dislocation loop. It is simply the same as the asymptotic behavior of the integrand. Since Green's tensor $G_{jn}(\mathbf{r})$ varies as r^{-1}, we see that $u_n^T(\mathbf{r})$ will vary as r^{-2} as $r \to \infty$. Since the strain $e_{mn}(\mathbf{r})$ varies as the derivative of the displacement, it will go as r^{-3} as $r \to \infty$. These results are listed in table 3. A more accurate calculation of the asymptotic displacement from a small loop can also be made from (3.30) by expanding Green's tensor as a Taylor series in $\mathbf{r}'$ for a few terms and integrating over S. If this result is specialized to isotropy, we find the same relations as were given by Kroupa [17]. The details will be shown in a future publication [26].

We now find the total distortion from (3.30)

$$\begin{aligned} u_{n,\,m}^T(\mathbf{r}) &= \int_S C_{ijkl} G_{jn,\,im}(\mathbf{R})\, b_l dS'_k \\ &= \oint_L \epsilon_{pmk} C_{ijkl} G_{jn,\,i}(\mathbf{R})\, b_l dL'_p + \int_S C_{ijkl} G_{jn,\,ik}(\mathbf{R})\, b_l dS'_m \\ &= \oint_L \epsilon_{pmk} C_{ijkl} G_{jn,\,i}(\mathbf{R})\, b_l dL'_p + \beta_{mn}^P(\mathbf{r}). \end{aligned} \tag{3.31}$$

Here the first equality follows by simply differentiating under the integral sign, where Green's tensor is the only function depending on $\mathbf{r}$, the second equality follows from Stokes' theorem (A4), and the third from (2.7), (B5), and (3.21). From (3.9) we then obtain the elastic distortion

$$\beta_{mn}(\mathbf{r}) = \oint_L \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})\, b_l dL'_p \tag{3.32}$$

This relation can of course also be obtained directly from (3.17), (3.23), and (B11). A similar result was derived by Mura [13] for a moving dislocation line.

We see that the state quantity β_{mn} can be written as a line integral along the discrete dislocation, i.e., it is expressed entirely in terms of an integral over the only regions of the body where the defect is localized. In general, we suggest that for a defect which is a state quantity, the necessary and sufficient condition for an associated field quantity to be a state quantity is that it can be written as an integral over the defect. For a line defect this means that the state quantity must be a line integral along the defect.

Equation (3.32) also allows us to estimate the asymptotic behavior of the distortion, and hence the strain, at large distances from a straight dislocation line. Due to the integration it is simply the same as that of Green's tensor, i.e., $\beta_{mn}(\mathbf{r})$ and $e_{mn}(\mathbf{r})$ vary as r^{-1} as $r \to \infty$. Since the displacement u_n^T is an integral of the distortion, it will vary as $\ln r$ as $r \to \infty$. These well-known results are also listed in table 3.

3.3. Continuous Distribution of Infinitesimal Dislocation Loops

In section 3.1 the dislocation density and the Burgers vector were defined by (3.4) and (3.7) in terms of the given plastic distortion. A consequence of these definitions is (3.8), relating the Burgers vector to the dislocation density. This equation could alternatively be used to define the dislocation density in terms of the Burgers vector, if this quantity is prescribed in a suitable manner. It is convenient to put this relation into differential form. When the dislocations are continuously distributed the density tensor is defined locally by

$$\alpha_{pl} \equiv \frac{\Delta b_l}{\Delta S_p}. \tag{3.33}$$

As we noted above, for a distribution of discrete dislocations this represents the average dislocation density, where Δb_l is the lth component of the resulting Burgers vectors of all the dislocations which pierce through a surface element ΔS_p oriented normal to the x_p direction at the given point.

In a similar way Kroupa [5] introduced the dislocation loop density tensor. He defined it as follows: γ_{kl} represents the flux of dislocation (or Burgers vector) in the x_l direction that encloses a unit vector in the x_k direction. When the loops are continuously distributed the density tensor γ_{kl} is a function of the position and is defined locally by

$$\gamma_{kl} \equiv \frac{\Delta b_l}{\Delta L_k}. \tag{3.34}$$

Again, for a distribution of discrete loops this represents the average dislocation loop density, where now Δb_l is the lth component of the resulting Burgers vectors of all the loops which are pierced by the line element ΔL_k oriented in the x_k direction at the given point.

To find the relation between the dislocation loop density and the plastic distortion, we first convert (3.34) to integral form

$$b_l = \oint_\lambda \gamma_{kl} dL_k. \tag{3.35}$$

Now we compare this relation to the definition (3.7). The integrands can only differ by a gradient with respect to x_k. Therefore we can set

$$\beta^P_{kl} = -\gamma_{kl} + u^P_{l,k}, \tag{3.36}$$

where u^P_l is an arbitrary vector field. This relation identifies the plastic distortion for a continuous distribution of dislocation loops. With it we can find all the relations derived in section 3.1 in terms of loops. For example, (3.4) leads to

$$\alpha_{pl} = \epsilon_{pmk}\, \gamma_{kl,m}. \tag{3.37}$$

This is the fundamental relation between the dislocation loop density and the corresponding dislocation density. Since a plastic displacement does not contribute to the elastic fields, (c.f. sec. 2.6), we can set $u^P_l = 0$, without loss of generality and so we can use

$$\beta^P_{kl} = -\gamma_{kl} \tag{3.38}$$

for the purpose of calculating the fields of a continuous distribution of dislocation loops. This agrees with Kroupa's identification. Hence we can identity the plastic distortion with the dislocation loop density, except for a minus sign.

We now also can give an interpretation of (3.21) in terms of infinitesimal loops. To construct a discrete dislocation line L, we distribute a constant density of infinitesimal dislocation loops of strength b_l over any surface S, whose boundary is L. This method could be taken as an alternative to the definition of the discrete dislocation line given at the beginning of section 3.2. So we conclude that for a finite dislocation loop the loop density is given by

$$\gamma_{kl}(\mathbf{r}) = \delta_k(S)\, b_l, \tag{3.39}$$

where S is a surface that spans the dislocation line. The vector $\delta_k(S)$ is the Dirac delta function on the surface S and it is always normal to S. In a plane we have a prismatic or a slip loop according to whether the Burgers vector is normal or parallel to S, respectively. Therefore (3.39) shows that the diagonal and off-diagonal components of γ_{kl} represent the prismatic and slip components of the dislocation loop density, respectively (see table 1). Equation (3.39) shows how to make the transition from a continuous distribution of infinitesimal loops to a finite loop. By (B15) we see that it satisfies (3.35), which also remains valid for many finite loops, and therefore can be used for the transition from finite loops to a continuous distribution: For many finite dislocation loops the average dislocation loop density γ_{kl} represents the x_l component of the sum of the Burgers vectors of all the loops whose surfaces are intersected by a unit vector in the x_k direction. For another interpretation of γ_{kl} consider the result

$$\begin{aligned}\int_V \gamma_{kl}(\mathbf{r})\, dV &= \int_V \delta_k(S)\, b_l dV \\ &= b_l \int \delta_k(S)\, \delta(V)\, dV \\ &= b_l \int_S \delta(V)\, dS_k \\ &= b_l \int_{S(V)} dS_k.\end{aligned} \tag{3.40}$$

Here the first equality follows from (3.39), the second from (B7), the third from (B12), and the fourth from (B7), where $S(V)$ is the part of the surface S inside V only. From this expression we see that the average dislocation loop density γ_{kl} also represents the sum of the x_l component of the Burgers vectors times the projected area on a plane normal to the x_k direction of all the dislocation loops per unit volume. The equivalence of the above two interpretations of the average dislocation loop density can also be shown by the methods of quantitative stereology [16].

3.4. The Dislocation Dipole

Kroupa [18] has also treated the fields of a dislocation dipole, i.e., a close pair of dislocations with opposite Burgers vectors. In this section we wish to present some general formulas for such a defect.

3.4.1. General Definition of a Dipole

We first give a very general definition of the dipole conjugate to any defect, following a similar line of reasoning as Kroupa used. The dipole is composed of two parts: the first is obtained from the basic defect by translating it through a small distance, and the second is the negative of the basic defect at its original position. To give this concept a mathematical formulation, let the basic defect be given as a function of position $\mathbf{r}$ by the source function $S(\mathbf{r})$. For example, this could be e^{P}_{kl} of section 2, or α_{pl} of section 3.1. If this defect is rigidly translated through a distance $\boldsymbol{\xi}$, the source function of the new defect configuration becomes $S(\mathbf{r}-\boldsymbol{\xi})$. Hence the source function of the conjugate dipole is given by

$$S^{D}(\mathbf{r})=S(\mathbf{r}-\boldsymbol{\xi})-S(\mathbf{r}). \tag{3.41}$$

This result applies to a finite dipole.

For a discrete defect concentrated on a point, line, or surface, it is customary to deal with the infinitesimal dipole. It is obtained by letting $\boldsymbol{\xi}$ approach zero and the strength of the basic defect approach infinity in such a way that the field of the dipole remains finite. If $\boldsymbol{\xi}$ is infinitesimal, we can use Taylor's expansion to write (3.41) as follows:

$$S^{D}(\mathbf{r})=-\xi_i S_{,i}(\mathbf{r}). \tag{3.42}$$

We see that our definition has the opposite sign from Kroupa's, but it agrees with the convention in electrodynamics. We shall now show that a similar relation holds between the fields of a defect and its conjugate dipole. Let the field of the basic defect be given by the generic expression:

$$f(\mathbf{r})=\int G(\mathbf{R})S(\mathbf{r}')dV', \tag{3.43}$$

where $G(\mathbf{R})$ is some kernel of integration. For example, (2.15) and (3.17) have this form. The corresponding field of the conjugate dipole is

$$\begin{aligned} f^{D}(\mathbf{r})&=\int G(\mathbf{R})S^{D}(\mathbf{r}')dV' \\ &=-\xi_i\int G(\mathbf{R})S_{,i'}(\mathbf{r}')dV' \\ &=-\xi_i\int G_{,i}(\mathbf{R})S(\mathbf{r}')dV' \\ &=-\xi_i f_{,i}(\mathbf{r}). \end{aligned} \tag{3.44}$$

Here the second equality follows from (3.42), the third by a partial integration, and the fourth from (3.43).

Equations (3.42) and (3.44) are the fundamental relations between functions of the basic defect and the corresponding function of its conjugate dipole. The basic defect here is arbitrary. It could for example be a dipole itself; in this case we obtain the dipole of a dipole, or a quadrupole. In this way all higher order multipoles are defined.

3.4.2. Application to the Discrete Dislocation Dipole

We now apply the above results to the case of a discrete dislocation line, figure 2. For example, the basic equation for the dislocation density is given by (3.23). Therefore the dislocation density

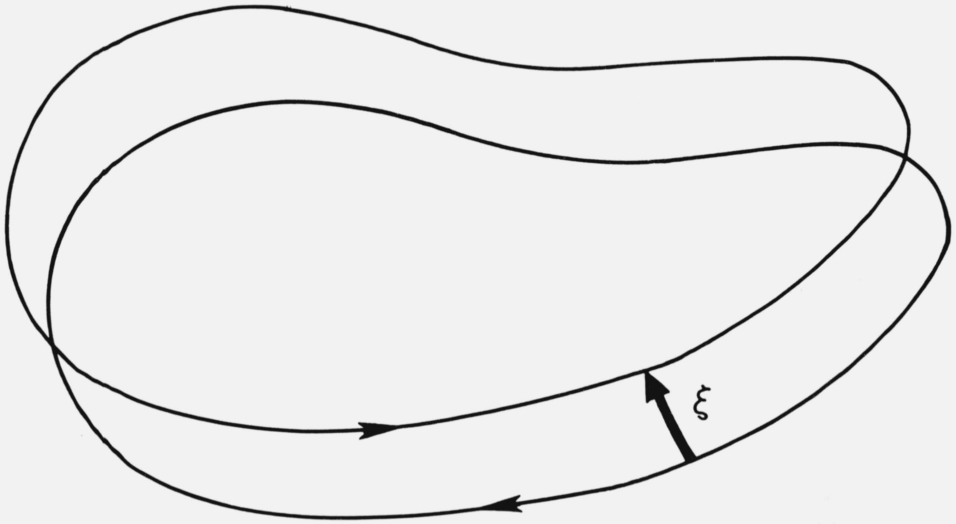

FIGURE 2. *The discrete defect dipole line conjugate to the basic defect line of figure 1.*

This figure shows the finite case. For the infinitesimal case $\xi \to 0$.

of the conjugate dislocation dipole is by (3.42):

$$\begin{aligned} \alpha^D_{pl}(\mathbf{r}) &= -\xi_m \alpha_{pl,\,m}(\mathbf{r}) \\ &= -\delta_{p,\,m}(L)\; b_l \xi_m. \end{aligned} \tag{3.45}$$

With this explicit expression we can clarify the meaning of the infinitesimal displacement ξ_m: We leg $\xi_m \to 0$ and $b_l \to \infty$ in such a way that $b_l \xi_m$ remains constant.

Next we find the displacement field of a dislocation dipole from (3.44)

$$\begin{aligned} u^D_n(\mathbf{r}) &= -\xi_m u^T_{n,\,m}(\mathbf{r}) \\ &= -\oint_L \epsilon_{pmk} C_{ijkl} G_{jn,\,i}(\mathbf{R})\; b_l \xi_m dL'_p + \delta_m(S)\; b_n \xi_m, \end{aligned} \tag{3.46}$$

where we have used (3.31) and (3.21). It is interesting to note that the last term in (3.46) can be made to vanish by choosing S to lie along $\boldsymbol{\xi}$, or $\xi_m dS'_m = 0$, i.e., by letting S be an infinitesimal strip connecting the basic dislocation to its displaced partner. As a consequence we see that the displacement of a dislocation dipole is a state quantity.

From (3.46) other static quantities can be derived, such as the distortion β^D_{mn}. The same result can alternatively be obtained by substituting (3.45) into (3.17).

Equation (3.46) also allows us to estimate the asymptotic behavior of the displacement at large distances from a straight dislocation dipole line. By comparison with (3.32) it is the same as that of the distortion of a straight dislocation line, i.e., it varies as r^{-1} as $r \to \infty$. Hence the strain will vary as r^{-2} as $r \to \infty$. These results are also listed in table 3. We note from the table that the dislocation dipole can be classified as a defect with properties between those of the dislocation line and the dislocation loop.

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Sometimes a narrow elongated dislocation loop is regarded as a dipole. This is not possible with our definition, since the two components of our dipole must be disjoint. However, two parallel, infinite, straight dislocations with opposite Burgers vectors are included in our definition of a dipole. This dipole resembles an elongated loop. By contrast, it is possible to regard a dislocation dipole as a special type of loop.

4. Continuous Distribution of Defects

As discussed in section 1.1, we shall denote the combination of dislocations and disclinations by the word *defects*, see table 2.

TABLE 2. *Generalization from dislocation theory to defect theory*

Quantity	Dislocation theory	Defect theory
Defect density tensors	Dislocation density $\boldsymbol{\alpha}$	Dislocation density $\boldsymbol{\alpha}$ Disclination density $\boldsymbol{\theta}$
Basic plastic fields	Plastic distortion $\boldsymbol{\beta}^P$	Plastic strain $\mathbf{e}^P$ Plastic bend-twist $\boldsymbol{\kappa}^P$
Characteristic vectors	Burgers vector $\mathbf{b}$	Total Burgers vector $\mathbf{B}$ Frank vector $\boldsymbol{\Omega}$
Basic elastic fields	Elastic distortion $\boldsymbol{\beta}$	Elastic strain $\mathbf{e}$ Elastic bend-twist $\boldsymbol{\kappa}$
Jump conditions	Displacement jump $[\mathbf{u}]$	Displacement jump $[\mathbf{u}]$ Rotation jump $[\boldsymbol{\omega}]$
Defect loop density tensors	Dislocation loop density $\boldsymbol{\gamma}(\boldsymbol{\beta}^P)$	Dislocation loop density $\boldsymbol{\gamma}(\boldsymbol{\beta}^*)$ Disclination loop density $\boldsymbol{\zeta}(\boldsymbol{\phi}^*)$

4.1. Geometry

Consider now an infinitely extended body in which the plastic strain e^P_{kl} and bend-twist κ^P_{kq} are given as independently prescribed functions of space. For convenience we refer to these two functions as the *basic plastic fields*. Then we may have disclinations as well as dislocations. The dislocation and disclination densities are defined by (I6.3, I6.1)

$$\alpha_{pl} \equiv -\epsilon_{pmk}(e^P_{kl,\,m} + \epsilon_{klq}\kappa^P_{mq}), \tag{4.1}$$

$$\theta_{pq} \equiv -\epsilon_{pmk}\kappa^P_{kq,\,m}, \qquad (\theta_{qp} = -\epsilon_{qnl}\kappa^P_{lp,\,n}). \tag{4.2}$$

The continuity equations (I6.8, I6.2)

$$\alpha_{pl,\,p} + \epsilon_{lpq}\theta_{pq} = 0, \tag{4.3}$$

$$\theta_{pq,\,p} = 0 \tag{4.4}$$

follow immediately from (4.1) and (4.2). The former means that dislocations can only end on disclinations, and conversely, if the disclination density is asymmetric, dislocations must emerge from it. The latter shows that disclinations cannot end inside the body.

Associated with the *Burgers circuit* λ we now define the *characteristic vectors*, the general Burgers vector B_l and the characteristic rotation vector Ω_q, which we have agreed to call the Frank vector in section 1.2, by

$$B_l \equiv -\oint_\lambda (e^P_{kl} - \epsilon_{lqr}\kappa^P_{kq}x_r)\, dL_k, \tag{4.5}$$

$$\Omega_q \equiv -\oint_\lambda \kappa^P_{kq} dL_k. \tag{4.6}$$

These relations can be interpreted as follows: Starting with a perfect crystal we can imagine that the plastic deformation is produced by letting defects migrate into the crystal. A number of them cut through λ. Every disclination that cuts through λ produces a relative rotation $-\kappa^P_{kq}dL_k$ in the lattice at the curve. These contributions added around the contour λ give the resultant Frank vector of all disclinations that remain stuck through the surface σ bounded by λ. The relation (4.5) is easily seen to be identical to (3.7), when κ^P_{kq} is given by (3.12), by doing a partial integration, where the integrated part vanishes, and using (3.1) or (3.2). Therefore, it is simply a generalization of (3.7). It represents the resultant Burgers vector of all defects (dislocations and disclinations) that have cut through λ and remain stuck through σ. We wish to emphasize that B_l does not represent the relative displacements added around the contour λ. The latter quantity is not an invariant for the curve λ, but depends on the point where the integration is started. The definition (4.5) is essentially motivated by Weingarten's theorem (section 5.1). By Stokes' theorem (A2) we have from (4.5–6), (I7.4, I7.3)

$$\begin{aligned} B_l &= -\int_\sigma \epsilon_{pmk}\,(e^P_{kl,\,m} - \epsilon_{lqm}\kappa^P_{kq} - \epsilon_{lqr}\kappa^P_{kq,\,m}x_r)\; dS_p \\ &= \int_\sigma (\alpha_{pl} - \epsilon_{lqr}\theta_{pq}x_r)\; dS_p, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \Omega_q &= -\int_\sigma \epsilon_{pmk}\kappa^P_{kq,\,m} dS_p \\ &= \int_\sigma \theta_{pq}\, dS_p, \end{aligned} \tag{4.8}$$

using (4.1) and (4.2). Relation (4.7) shows that the disclination density θ_{pq} also contributes to the general Burgers vector, in addition to the dislocation density α_{pl}. Relation (4.8) shows that the disclination density θ_{pq} represents the flux of disclination (or Frank vector) in the x_q direction that crosses unit area of a plane normal to the x_p direction.

For later application we also write (4.1–2) in their equivalent forms

$$\epsilon_{pmk}\alpha_{pl} \equiv e^P_{ml,\,k} - e^P_{kl,\,m} + \epsilon_{pmk}\kappa^P_{lp} - \epsilon_{klm}\kappa^P, \tag{4.9}$$

$$\epsilon_{pmk}\theta_{pq} \equiv \kappa^P_{mq,\,k} - \kappa^P_{kq,\,m}\,,\ (\epsilon_{qnl}\theta_{qp} = \kappa^P_{np,\,l} - \kappa^P_{lp,\,n}). \tag{4.10}$$

The incompatibility tensor is defined by (I4.1, I6.6)

$$\begin{aligned} \eta_{pq} &\equiv \epsilon_{pmk}\epsilon_{qnl}e^P_{kl,\,mn} \qquad (4.11) \\ &= -\,(\epsilon_{qnl}\alpha_{pl,\,n} + \theta_{pq})_{(pq)}, \qquad (4.12) \end{aligned}$$

where the second equality follows from (4.1–2). The continuity equation for the compatibility (I4.2),

$$\eta_{pq,p} = 0, \tag{4.13}$$

follows immediately from (4.11).

With disclinations we claim that the total distortion is no longer the sum of an elastic and plastic part, (3.9), simply because β^P_{mn} is not defined. For, if β^P_{mn} existed, the plastic bend-twist would be the gradient of the plastic rotation, (3.12), and consequently the disclination density (4.1) would vanish.[10] Instead, since the plastic strain e^P_{kl} and bend-twist κ^P_{kq} are prescribed, we postulate the existence of the elastic strain e_{kl} and bend-twist κ_{kq}, which are called the *basic elastic fields*. So we have (I2.3, I4.3, I2.16, and I5.17):

$$e^T_{mn} \equiv u^T_{(n,m)} = e_{mn} + e^P_{mn}, \tag{4.14}$$

$$\kappa^T_{st} \equiv \omega^T_{t,s} = 1/2\, \epsilon_{tmn}\, u^T_{n,ms} = \kappa_{st} + \kappa^P_{st}. \tag{4.15}$$

Since it follows from (4.14–15) that

$$\epsilon_{pmk}\, (e^T_{kl,m} + \epsilon_{klq}\, \kappa^T_{mq}) = \epsilon_{pmk}\, u^T_{l,km} = 0, \tag{4.16}$$

these relations allow us to derive the following basic geometric laws or field equations for α_{pl} and θ_{pq} from (4.1–2), (I6.11, I6.10)

$$\epsilon_{pmk}\, (e_{kl,m} + \epsilon_{klq}\, \kappa_{mq}) = \alpha_{pl}, \tag{4.17}$$

$$\epsilon_{pmk}\, \kappa_{kq,m} = \theta_{pq}. \tag{4.18}$$

The geometric meaning of these equations is that the defects (dislocations and disclinations) are the sources of the basic elastic fields (elastic strain and bend-twist). It also follows from (4.14–15) that

$$e^T_{kl} - \epsilon_{lqr}\, \kappa^T_{kq}\, x_r = (u^T_l - u^T_{[l,r]}\, x_r)_{,k}, \tag{4.19}$$

and hence these relations also allow us to derive the basic geometric laws for B_l and Ω_q from (4.5–6), (I7.2, I7.1)

$$\oint_\lambda (e_{kl} - \epsilon_{lqr}\, \kappa_{kq}\, x_r)\, dL_k = B_l, \tag{4.20}$$

$$\oint_\lambda \kappa_{kq}\, dL_k = \Omega_q. \tag{4.21}$$

These may be regarded as the field equations in integral form, equivalent to (4.17–18). Finally, the relation (4.14) allows us to derive the basic geometric law or field equation for the incompatibility η_{pq} from (4.11), (I 4.4)

$$-\epsilon_{pmk}\, \epsilon_{qnl}\, e_{kl,mn} = \eta_{pq}, \tag{4.22}$$

which shows that we can isolate the incompatibility as the source of the elastic strain.

4.2. Statics

This section extends the work of reference [3], which gave only the geometry of a continuous distribution of defects, reviewed in section 4.1. The main result we shall find is an explicit and new expression for the elastic strain, eq (4.29), in terms of defect densities. We shall also find the elastic bend-twist.

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[10] This situation is analogous to the case of pure dislocations: There the total displacement is no longer the sum of an elastic and plastic part, $u^T_l = u_l + u^P_l$. For, if u^P_l existed, the plastic distortion would be a gradient, $\beta^P_{kl} = u^P_{l,k}$, and the dislocation density (3.4) would vanish.

We find the total distortion from (2.15)

$$
\begin{aligned}
u^T_{n,m}(\mathbf{r}) &= -\int C_{ijkl} G_{jn,im}(\mathbf{R}) e^P_{kl}(\mathbf{r}')dV' \\
&= -\int C_{ijkl} G_{jn,i}(\mathbf{R}) e^P_{kl,m'}(\mathbf{r}')dV' \\
&= \int C_{ijkl} G_{jn,i}(\mathbf{R})[\epsilon_{pmk}\alpha_{pl}(\mathbf{r}') - e^P_{ml,k'}(\mathbf{r}') - \epsilon_{pmk}\kappa^P_{lp}(\mathbf{r}')]dV' \\
&= \int \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})[\alpha_{pl}(\mathbf{r}') - \kappa^P_{lp}(\mathbf{r}')]dV' + e^P_{mn}(\mathbf{r}).
\end{aligned} \tag{4.23}
$$

Here the second equality follows by a partial integration, the third from (4.9) and (2.3), and the fourth by a partial integration, (2.7) and (B3). We see that for the integral to converge it is only necessary that $\alpha_{pl}(\mathbf{r})$ and $\kappa^P_{lp}(\mathbf{r})$ vanish faster than r^{-1} as $r \to \infty$. From this relation we now proceed to derive the basic elastic fields and show that they are state quantities.

The elastic strain is found from (4.14)[11]

$$
e_{mn}(\mathbf{r}) = \int \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})[\alpha_{pl}(\mathbf{r}') - \kappa^P_{lp}(\mathbf{r}')]dV'_{(mn)}. \tag{4.24}
$$

To show this is a state quantity we shall use the concept of an incompatibility source tensor, introduced by Simmons and Bullough [12]. In contrast to Simmons and Bullough, who derived several forms of it from a general definition, we define it directly as follows

$$
I_{mnpq}(\mathbf{r}) \equiv (4\pi)^{-1} \int \epsilon_{pmk}\epsilon_{qsl} C_{ijkl} G_{jn,i's'}(\mathbf{r}') R^{-1} dV'_{(mn)}. \tag{4.25}
$$

We see from this relation that the homogeneity of $I_{mnpq}(\mathbf{r})$ is the same as that of $G_{jn}(\mathbf{r})$, i.e., of degree (-1), so that they both vary as r^{-1}. By using the identity

$$
\epsilon_{pmk}\epsilon_{qsl} \equiv \begin{vmatrix} \delta_{pq} & \delta_{ps} & \delta_{pl} \\ \delta_{mq} & \delta_{ms} & \delta_{ml} \\ \delta_{kq} & \delta_{ks} & \delta_{kl} \end{vmatrix} \tag{4.26}
$$

and (2.7) it can also be written as

$$
\begin{aligned}
I_{mnpq}(\mathbf{r}) = (4\pi)^{-1}[(\delta_{mn}\delta_{pq} - \delta_{mq}\delta_{np})r^{-1} - (C_{ijpq} - C_{ijkl}\delta_{pq}) \int G_{jn,i'm'}(\mathbf{r}')R^{-1}dV' \\
+ (C_{ijmq} - C_{ijkk}\delta_{mq}) \int G_{jn,i'p'}(\mathbf{r}')R^{-1}dV']_{(mn)}.
\end{aligned} \tag{4.27}
$$

Except for the last line, this expression agrees with the one Simmons and Bullough have called the Eshelby-Eddington formula. Now the incompatibility source tensor satisfies the following relationship:

$$
\begin{aligned}
\epsilon_{qsl} I_{mnpq,s}(\mathbf{r}) &= (4\pi)^{-1} \int \epsilon_{pmk}[C_{ijkl} G_{jn,i's'}(\mathbf{r}') - C_{ijks} G_{jn,i'l'}(\mathbf{r}')]R^{-1}_{,s} dV'_{(mn)} \\
&= (4\pi)^{-1} \int \epsilon_{pmk}[C_{ijkl} G_{jn,i'}(\mathbf{r}')R^{-1}_{,ss} - C_{ijks} G_{jn,i's'}(\mathbf{r}')R^{-1}_{,l}] dV'_{(mn)} \\
&= -\int \epsilon_{pmk}[C_{ijkl} G_{jn,i'}(\mathbf{r}')\delta(R) - \delta_{kn}\delta(\mathbf{r}')(4\pi R)^{-1}_{,l}] dV'_{(mn)} \\
&= -[\epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{r})]_{(mn)}.
\end{aligned} \tag{4.28}
$$

---

[11] See footnote 6 on page 53.

Here the second equality follows by partial integrations, and the third from $R^{-1}_{,ss}=-4\pi\delta(R)$ and (2.7). From the above the term in (4.24) containing κ^P_{lp} becomes

$$\int \epsilon_{qsl} I_{mnpq,s}(\mathbf{R})\kappa^P_{lp}(\mathbf{r}')\,dV' = \int \epsilon_{qsl} I_{mnpq}(\mathbf{R})\kappa^P_{lp,s'}(\mathbf{r}')\,dV'$$
$$= -\int I_{mnpq}(\mathbf{R})\theta_{qp}(\mathbf{r}')\,dV'$$

by a partial integration and (4.2). Note that in the partial integration the surface integral vanishes because of the asymptotic behavior of I_{mnpq} and κ^P_{lp}, discussed above. Hence we find for (4.24)

$$e_{mn}(\mathbf{r}) = \int \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})\,\alpha_{pl}(\mathbf{r}')\,dV'_{(mn)} - \int I_{mnpq}(\mathbf{R})\theta_{qp}(\mathbf{r}')\,dV'. \tag{4.29}$$

This is the result we seek. We see it is a state quantity because it depends entirely on the defect quantities α_{pl} and θ_{qp}.

The incompatibility source tensor I_{mnpq} was originally introduced by Simmons and Bullough to solve the so-called incompatibility problem, i.e., to find the elastic strain e_{mn} when the incompatibility η_{pq} is given as a prescribed function of space. We show here how this can be done. From (4.28)

$$\epsilon_{prk}\epsilon_{qsl} I_{mnpq,rs}(\mathbf{r}) = -[C_{ijkl}G_{jn,im}(\mathbf{r}) - \delta_{km}C_{ijrl}G_{jn,ir}(\mathbf{r})]_{(mn)}$$
$$= -[C_{ijkl}G_{jn,im}(\mathbf{r}) + \delta_{km}\delta_{ln}\delta(\mathbf{r})]_{(mn)}, \tag{4.30}$$

using (2.7). Therefore we find

$$\begin{aligned} e_{mn}(\mathbf{r}) &= -\int C_{ijkl}G_{jn,im}(\mathbf{R})e^P_{kl}(\mathbf{r}')\,dV'_{(mn)} - e^P_{mn}(\mathbf{r}) \\ &= \int [\epsilon_{prk}\epsilon_{qsl} I_{mnpq,rs}(\mathbf{R}) + \delta_{km}\delta_{ln}\delta(\mathbf{R})]\, e^P_{kl}(\mathbf{r}')\,dV' - e^P_{mn}(\mathbf{r}) \\ &= \int \epsilon_{prk}\epsilon_{qsl} I_{mnpq}(\mathbf{R})e^P_{kl,r's'}(\mathbf{r}')\,dV' \\ &= \int I_{mnpq}(\mathbf{R})\eta_{pq}(\mathbf{r}')\,dV'. \end{aligned} \tag{4.31}$$

Here the first equality follows from (4.14) and (2.15), the second from (4.30), the third from a partial integration, and the fourth from (4.11).

We next wish to derive the elastic bend-twist. First we find the derivative of the total distortion (4.23)

$$u^T_{n,ms}(\mathbf{r}) = \int \epsilon_{pmk} C_{ijkl} G_{jn,is}(\mathbf{R})[\alpha_{pl}(\mathbf{r}') - \kappa^P_{lp}(\mathbf{r}')]dV' + e^P_{mn,s}(\mathbf{r}). \tag{4.32}$$

The κ^P_{lp} term in this expression becomes by a partial integration

$$\begin{aligned} &-\int \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})\kappa^P_{lp,s'}(\mathbf{r}')\,dV' \\ &= \int \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})[\epsilon_{qsl}\theta_{qp}(\mathbf{r}') - \kappa^P_{sp,l'}(\mathbf{r}')]dV' \\ &= \int \epsilon_{pmk}\epsilon_{qsl} C_{ijkl} G_{jn,i}(\mathbf{R})\theta_{qp}(\mathbf{r}')\,dV' + \epsilon_{pmn}\kappa^P_{sp}(\mathbf{r}). \end{aligned}$$

Here the first equality follows from (4.10), and the second by a partial integration and (2.7). Hence

$$u^T_{n,ms}(\mathbf{r}) = \int \epsilon_{pmk} C_{ijkl} G_{jn,is}(\mathbf{R})\alpha_{pl}(\mathbf{r}')\,dV' + \int \epsilon_{pmk}\epsilon_{qsl} C_{ijkl} G_{jn,i}(\mathbf{R})\theta_{qp}(\mathbf{r}')\,dV'$$
$$+ e^P_{mn,s}(\mathbf{r}) + \epsilon_{pmn}\kappa^P_{sp}(\mathbf{r}). \tag{4.33}$$

Finally we have from (4.15)

$$\kappa_{st}(\mathbf{r})=1/2\int \epsilon_{tmn}\epsilon_{pmk}C_{ijkl}G_{jn,is}(\mathbf{R})\alpha_{pl}(\mathbf{r}')dV'+1/2\int \epsilon_{tmn}\epsilon_{pmk}\epsilon_{qsl}C_{ijkl}G_{jn,i}(\mathbf{R})\theta_{qp}(\mathbf{r}')dV', \tag{4.34}$$

which is seen to be a state quantity too.

This section then has extended the results for dislocation theory of section 3.1.2 to a more general defect theory. The central result there, eq (3.17), has been replaced by (4.29) and (4.34). We note that concepts, quantities, or equations from dislocation theory generalize into pairs of concepts, quantities, or equations in defect theory. Some of these ideas have been summarized in table 2.

5. The Discrete Defect Line

5.1. Weingarten's Theorem

The point of departure for the discrete defect line is the following theorem [3]:

WEINGARTEN'S THEOREM: *On following around an irreducible circuit in a multiply-connected body, the displacement and rotation change by an amount that represents a rigid body motion, if and only if the classical elastic compatibility conditions are satisfied throughout the body.*

Explicitly these changes are given by (I3.4, I3.3).

$$[u_l]=B_l+\epsilon_{lqr}\Omega_q x_r, \tag{5.1}$$

$$[\omega_q]=\Omega_q, \tag{5.2}$$

where the constants B_l and Ω_q are given by line integrals along the irreducible circuit λ (I3.6, I3.5):

$$B_l\equiv\oint_\lambda (e_{kl}-\epsilon_{lqr}\kappa_{kq}x_r)dL_k, \tag{5.3}$$

$$\Omega_q\equiv\oint_\lambda \kappa_{kq}dL_k. \tag{5.4}$$

That these quantities are constant is easily shown by Stokes' theorem and the compatibility equations, i.e. (4.17–18) with $\alpha_{pl}=0$, $\theta_{pq}=0$. We note incidentally that the definitions (5.3–4) are consistent with the relations (4.20–21).[12]

5.2. Geometry

The discrete defect line L is defined as the boundary of a surface S, where the material below S has been plastically displaced with respect to the material above S by an amount which represents a rigid motion (fig. 1).

Hence, the difference between the displacement just below and above S is given by

$$[u_l(\mathbf{r})]=b_l+\epsilon_{lqr}\Omega_q(x_r-x_r^0), \tag{5.5}$$

where b_l represents a rigid translation and the second term a superposed constant rotation of amount Ω_q around an axis through x_r^0. The constant b_l will be called the Burgers vector for the discrete dislocation line contained in the defect line, and is to be distinguished from the general

[12] For the nonlinear generalization of the present theory it would be necessary to determine if Weingarten's theorem still holds. Then for a finite rotation (5.1) would have to be modified into $[u_l]=B_l+(\epsilon_{lqr}\Omega_q x_r+\frac{1}{2}\Omega_l\Omega_q x_q-\frac{1}{2}\Omega^2 x_l)(1+\frac{1}{4}\Omega^2)^{-1}$. Here the Frank vector or versor has the direction of the rotation axis and the magnitude $2\tan\frac{1}{2}\Phi$, where Φ is the angle of rotation. The addition rule becomes $\Omega_q=(\Omega_q^{(1)}+\Omega_q^{(2)}-\frac{1}{2}\epsilon_{lqr}\Omega_r^{(1)}\Omega_l^{(2)})(1-\frac{1}{4}\Omega_p^{(1)}\Omega_p^{(2)})^{-1}$ for a rotation (1) followed by a rotation (2).

Burgers vector defined by (4.5). The constant Ω_q will be identified with the Frank vector (4.6). The relation (5.5) implies there is also a jump in rotation across S given by

$$[\omega_q(\mathbf{r})] = 1/2\epsilon_{klq}[u_l(\mathbf{r})]_{,k} = \Omega_q. \tag{5.6}$$

If we visualize the material in a tube around L removed (fig. 1), we have a doubly connected body, and since the jump across S represents a rigid body motion, Weingarten's theorem applies. Hence, this doubly connected body is compatible, i.e., the basic elastic fields satisfy the compatibility equations, even on S. Therefore, the results (5.5–6) should also follow from (5.1–4). The real, incompatible, simply connected body is obtained by letting the cross section of the tube vanish. Then (5.3–4) become identical to (4.20–21), and by the compatibility of the total deformation these relations are equivalent to (4.5–6).

Our problem now is how to embody the statements (5.5–6) into definitions for the basic plastic fields, i.e., the plastic strain and bend-twist. We give a straightforward operational procedure to obtain these quantities and then verify that they are correct by (5.1–2) and (4.5–6). Assume first that S is closed, enclosing the volume V. Then by (B7)

$$u_l^T(\mathbf{r}) = \delta(V)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\} \tag{5.7}$$

represents a displacement, which is the same as (5.5) inside V and vanishes outside V. Thus, it has the required jump across S. Equation (5.7) could be regarded as describing a grain of volume V and boundary S, whose orientation with respect to the surrounding material is given by the rigid motion (5.5). We assume that the deformation (5.7) comes about by means of a plastic deformation on the surface S and an elastic translation and rotation in V. To find the basic plastic fields we just calculate the basic total fields, i.e., the total strain and bend-twist. In general these can be split into elastic and plastic parts, (4.14–15). But since the only elastic deformation is a rigid motion, the basic elastic fields will vanish and the plastic fields will equal the total fields. We have from (3.9) and (5.7)

$$\begin{aligned}\beta_{kl}^T(\mathbf{r}) &= \delta_{,k}(V)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\} + \delta(V)\epsilon_{lqk}\Omega_q \\ &= -\delta_k(S)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\} + \delta(V)\epsilon_{klq}\Omega_q,\end{aligned} \tag{5.8}$$

using the divergence theorem (B24). From this we find

$$e_{kl}^T(\mathbf{r}) = -\delta_k(S)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\}_{(kl)}, \tag{5.9}$$

$$\omega_q^T(\mathbf{r}) = -1/2\,\epsilon_{klq}\delta_k(S)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\} + \delta(V)\Omega_q. \tag{5.10}$$

We see that (5.10) represents a rotation, which is the same inside V as (5.6) and vanishes outside V. Thus, it has the required jump across S. We next find the bend-twist from (4.15)

$$\kappa_{mq}^T(\mathbf{r}) = -1/2\,\epsilon_{klq}[\delta_k(S)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\}]_{,m} - \delta_m(S)\Omega_q, \tag{5.11}$$

using the divergence theorem again. We see that the basic total fields are concentrated at the surface S. As mentioned before, since the deformation is just a rigid motion of part of the body, there are no basic elastic fields, and therefore the fields are all plastic. The next step is to assume that (5.9) and (5.11) hold in the same form for the open surface S of the defect loop. To write down the final results it is convenient to introduce the "plastic distortion" and "plastic rotation" defined by Mura [4]:

$$\beta_{kl}^*(\mathbf{r}) \equiv -\delta_k(S)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\}, \tag{5.12}$$

$$\phi_{kq}^*(\mathbf{r}) \equiv -\delta_k(S)\Omega_q. \tag{5.13}$$

These quantities will be interpreted in section 6 as the dislocation and disclination loop densities. Until then they serve as convenient intermediate quantities for the purpose of calculation. Now we write

$$e^P_{kl}=\beta^*_{(kl)}, \tag{5.14}$$

$$\kappa^P_{mq}=1/2\epsilon_{klq}\beta^*_{kl,\,m}+\phi^*_{mq},$$

$$(\kappa^P_{kq}=1/2\epsilon_{lqr}\beta^*_{rl,\,k}+\phi^*_{kq}). \tag{5.15}$$

These are the results we looked for. Equations (5.14–15) together with (5.12–13) represent the basic plastic fields for a discrete defect line L spanned by the surface S. From (5.14–15) we also have the result

$$e^P_{kl,\,m}+\epsilon_{klq}\kappa^P_{mq}=\beta^*_{kl,\,m}+\epsilon_{klq}\phi^*_{mq}, \tag{5.16}$$

which will be useful for later purposes. As we mentioned above, we now check the validity of results (5.14–15). From (4.5)

$$B_l=-\oint_\lambda(\beta^*_{(kl)}-\beta^*_{[rl],\,k}x_r-\epsilon_{lqr}\phi^*_{kq}x_r)dL_k$$

$$=-\oint_\lambda(\beta^*_{kl}-\epsilon_{lqr}\phi^*_{kq}x_r)dL_k$$

$$=\oint_\lambda\delta_k(S)\{b_l-\epsilon_{lqr}\Omega_qx^0_r\}dL_k$$

$$=b_l-\epsilon_{lqr}\Omega_qx^0_r, \tag{5.17}$$

by a partial integration, (5.12–13), and (B15). This then is the relation between the general Burgers vector and the dislocation Burgers vector. Next we have by (5.15), (5.13), and (B15)

$$-\oint_\lambda\kappa^P_{kq}dL_k=-\oint_\lambda\phi^*_{kq}\,dL_k$$

$$=\oint_\lambda\delta_k(S)\Omega_qdL_k$$

$$=\Omega_q. \tag{5.18}$$

From (4.6) this relation identifies Ω_q as the Frank vector. If we substitute these results into (5.1–2) we obtain (5.5–6), as was required.

We now find the dislocation density from (4.1) and (5.16)

$$\alpha_{pl}(\mathbf{r})=-\epsilon_{pmk}(\beta^*_{kl,\,m}+\epsilon_{klq}\phi^*_{mq})$$

$$=\epsilon_{pmk}[\delta_{k,\,m}(S)\{b_l+\epsilon_{lqr}\Omega_q(x_r-x^0_r)\}+\delta_k(S)\epsilon_{lqr}\Omega_q\delta_{rm}+\epsilon_{klq}\delta_m(S)\Omega_q]$$

$$=\delta_p(L)\{b_l+\epsilon_{lqr}\Omega_q(x_r-x^0_r)\}, \tag{5.19}$$

where the first equality follows from (5.16), the second from (5.12–13), and the third by Stokes' theorem (B26) and a cancellation. The disclination density is obtained from (4.2)

$$\begin{aligned}\theta_{pq}(\mathbf{r}) &= -\epsilon_{pmk}\phi^*{}_{kq,\,m} \\ &= \epsilon_{pmk}\delta_{k,\,m}(S)\Omega_q \\ &= \delta_p(L)\Omega_q \end{aligned} \tag{5.20}$$

where we have used (5.15), (5.13), and (B26). Relations (5.19–20) represent the defect densities for a discrete defect line. We see from (5.19) that the Frank vector Ω_q also contributes to the dislocation density, in addition to the dislocation Burgers vector b_l. In these relations L is the closed boundary of S. The vector $\delta_p(L)$ is the Dirac delta function on the curve L and it is always parallel to L. A discrete disclination line is called wedge or twist when the Frank vector is parallel or normal to the line, respectively. Therefore, (5.20) shows that the diagonal and off-diagonal components of θ_{pq} represent the wedge and twist components of the disclination density, respectively (see table 1). Equations (5.19–20) show how to make the transition from a continuous distribution of defects to a discrete defect line.

The discrete defect line, characterized by the dislocation Burgers vector b_l and the Frank vector Ω_q, was defined in this section independent of the defect density tensors α_{pl} and θ_{pq} defined by (4.1–2) in section 4.1. The question arises whether there is a unique correlation between these definitions. For dislocations only there is a straightforward relation between the Burgers vector and the dislocation density of a continuous distribution, given by (3.8), or between the dislocation density for a discrete line and its Burgers vector, given by (3.23). When disclinations are introduced there is a similar straightforward relation between the Frank vector and the disclination density of a continuous distribution, given by (4.8), or between the disclination density for a discrete line and its Frank vector, given by (5.20). However, as we noted, the general Burgers vector now contains a contribution from the disclination density, eq (4.7), or the dislocation density for a discrete defect line contains a contribution from the Frank vector, eq (5.19). What this means is that the definitions of the "dislocation" are not identical in both approaches. Therefore, as we see from the relations quoted, in the transition from one formulation to the other a certain amount of mixing occurs. Anthony [2] has handled this difficulty by adopting the discrete disclination loop as the true definition of the disclination. Then our dislocation density tensor α_{pl} in (5.19) partly describes the discrete disclination. He therefore divides α_{pl} into two parts, a true component that corresponds to the dislocation line with Burgers vector b_l, and a component that belongs to the disclination line with Frank vector Ω_q. Hence our difference with Anthony merely reduces to a difference in point of view.

We prefer to retain our own point of view with different definitions of the "dislocation" because it is in fact difficult to identify the dislocation line in a discrete defect line, as will be seen from (5.19). For example, let us change the axis of rotation and the dislocation Burgers vector to

$$x_r^{o\prime} = x_r^0 + \xi_r, \tag{5.21}$$

$$b_l' = b_l + \epsilon_{lqr}\Omega_q\xi_r. \tag{5.22}$$

Then α_{pl} in (5.19) is unaltered. So the dislocation Burgers vector b_l is not uniquely defined, but depends on the location of the axis. On the other hand, the general Burgers vector B_l in (5.17) is unaltered by (5.21–22), and, therefore this is the quantity which is invariant for a discrete defect loop. This is another motivation for introducing it.

The significance of (5.19) can further be illustrated as follows. Consider an infinitesimal volume dV centered about some point on L. The jump across S is given by (5.5). The rotational part of (5.5) can be approximated by a constant inside dV, since it is so small, i.e., locally we cannot determine if the displacement jump is due to a rotation or a translation, even if we know the rotation jump locally. Now the dislocation density is a local tensor field. Therefore the dislocation density tensor can be found by replacing b_l in (3.23) by the jump (5.5), in agreement with (5.19). Hence, the dislocation density (5.19) at a point on the line L is exactly what would be expected on the basis of the local plastic displacement near the point, or a small Burgers circuit around L at the point. We shall illustrate this point more explicitly by examples in future publications [25, 26].

From (5.21–22) it would seem at first that we could eliminate the discrete dislocation from (5.19), by choosing ξ_r such that $b_l=0$, but this is only possible if b_l is normal to Ω_q. However, we can draw the following important conclusion from (5.19). The axis of a discrete disclination line can be translated from the point x_r^0 to the point $x_r^{0\prime}$ by adding a discrete dislocation to the line with a Burgers vector given by

$$b_l=\epsilon_{lqr}\Omega_q(x_r^0-x_r^{0\prime}), \tag{5.23}$$

which is normal to the Frank vector Ω_q. In other words, we can move the axis by adding the dislocation density

$$\alpha_{pl}(\mathbf{r})=\delta_p(L)\epsilon_{lqr}\Omega_q(x_r^0-x_r^{0\prime}), \tag{5.24}$$

as is evident from (5.19).

The above development suggests the possibility that a dislocation could end on or originate from a disclination line. Consider the three curves L, L', and L'', illustrated in figure 3, which

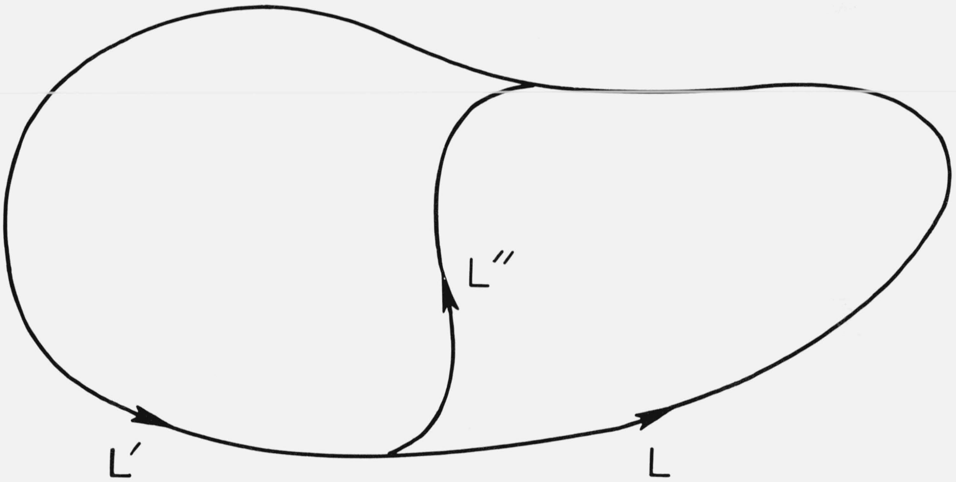

FIGURE 3. *Defect lines which join at nodes.*

meet at nodes. These curves could represent the discrete defect lines described by the following expressions:

$$\alpha_{pl}(\mathbf{r})=\delta_p(L)\{b_l+\epsilon_{lqr}\Omega_q(x_r-x_r^0)\}+\delta_p(L')\{b_l+\epsilon_{lqr}\Omega_q(x_r-x_r^{0\prime})\}$$

$$+\delta_p(L'')\epsilon_{lqr}\Omega_q(x_r^0-x_r^{0\prime}), \tag{5.25}$$

$$\theta_{pq}(\mathbf{r})=\delta_p(L+L')\Omega_q. \tag{5.26}$$

There are several ways in which one can view this defect. First, it consists of a defect line as described by (5.19–20) along the curve L and L' to which a dislocation line with Burgers vector (5.23) has been added along L' and L''. Second, it is a dislocation line along L'', which connects two points of a defect line along L and L'; as a consequence the axis goes thru x_r^0 for L and thru $x_r^{0\prime}$ for L'. Third, it consists of two defect loops with the same Frank vector, one along L' and L''

with axis thru $x_r^{0\prime}$, and one along L and minus L'' with axis thru x_r^0; along the line of overlap L'' the rotational parts cancel and only a dislocation line is left. So we see how the defect described by the relations (5.25–26) can be regarded as a combination of two simpler defects. We also see here how a discrete dislocation line can end on a discrete disclination line. A special case of this geometry has already been discussed [19] and a detailed mathematical analysis of this particular example will be presented in a future publication [25].

We note that (5.19–20) satisfy the continuity equations (4.3–4) by (B28):

$$\alpha_{pl,p}+\epsilon_{lpq}\theta_{pq}=\delta_p(L)\epsilon_{lqr}\Omega_q\delta_{rp}+\epsilon_{lpq}\delta_p(L)\Omega_q=0, \tag{5.27}$$

$$\theta_{pq,p}=\delta_{p,p}(L)\Omega_q=0, \tag{5.28}$$

Similarly, it is easy to show that (5.25–26) also satisfy the continuity equations, confirming an assertion we have made [19].

As a cross-check we also show that (5.19–20) give consistent results for (4.7–8)

$$B_l=\int_\sigma \delta_p(L)\{b_l-\epsilon_{lqr}\Omega_q x_r^\circ\}dS_p=b_l-\epsilon_{lqr}\Omega_q x_r^\circ, \tag{5.29}$$

$$\int_\sigma \theta_{pq}dS_p=\int_\sigma \delta_p(L)\Omega_q dS_p=\Omega_q, \tag{5.30}$$

by (B15), in agreement with (5.17–18). These relations remain valid for many defect lines as well, and could therefore be used to show how to make the transition from discrete lines to a continuous distribution of defects. However, the following relations are more convenient to make this point:

$$\begin{aligned}\int_\sigma \alpha_{pl}dS_p&=\int_\sigma \delta_p(L)\{b_l+\epsilon_{lqr}\Omega_q(x_r-x_r^\circ)\}dL_p\\ &=b_l-\epsilon_{lqr}\Omega_q(x_r^{L\sigma}-x_r^\circ)\\ &=[u_l]^{L\sigma},\end{aligned} \tag{5.31}$$

$$\int_\sigma \theta_{pq}dS_p=[\omega_q]\cdot \tag{5.32}$$

Here the first equality follows from (5.19), the second from (B16) where $x_r^{L\sigma}$ is the point of intersection of the curve L with the surface σ (fig. 1), the third from (5.5), and the last from (5.6). So for many defect lines the average dislocation density α_{pl} represents the x_l component of the sum of the displacement jump vectors and the average disclination density θ_{pq} the x_q component of the sum of the rotation jump (or Frank) vectors of all the defect lines that intersect unit area of a plane normal to the x_p direction.

5.3. Statics

In this section we find the basic elastic fields, i.e., the elastic strain and bend-twist, for a discrete defect line.

If we substitute (5.14) and (5.12) into (2.15) we find the displacement

$$\begin{aligned}u_n^T(\mathbf{r})&=\int C_{ijkl}G_{jn,i}(\mathbf{R})\delta_k(S')\{b_l+\epsilon_{lqr}\Omega_q(x_r'-x_r^0)\}dV'\\ &=\int_S C_{ijkl}G_{jn,i}(\mathbf{R})\{b_l+\epsilon_{lqr}\Omega_q(x_r'-x_r^0)\}dS_k'\end{aligned} \tag{5.33}$$

by (B12). This is the expression for the total displacement due to a finite discrete defect loop. It allows us to estimate the asymptotic behavior of the displacement at large distances from a small disclination loop, $b_l=0$. Since Green's tensor $G_{jn}(\mathbf{r})$ varies as r^{-1}, we see that $u_n^T(\mathbf{r})$ will in general vary as r^{-2} as $r\to\infty$. However, for special cases such as a symmetric loop centered on its axis, the integral vanishes to first order by symmetry. Therefore a finite symmetric disclination loop has the anomalous asymptotic dependence of r^{-3} as $r\to\infty$. Since the strain $e_{mn}(\mathbf{r})$ is obtained from the derivative of the displacement, it will vary as r^{-4} as $r\to\infty$. These results are listed in table 3.

TABLE 3

Asymptotic behavior of the displacement $\mathbf{u}^T$ and the strain $\mathbf{e}$ at large distances from certain defect configurations. The numbers in parentheses refer to the equations from which the estimate is obtained. A symmetric disclination loop is one for which $\int_S (x'_r-x^0_r)dS'_k=0$.

	Dislocation		Disclination	
	u^T	e	u^T	e
Line	$\ln r$	r^{-1} (3.32)	$r\ln r$	$\ln r$ (5.37)
Dipole	r^{-1} (3.46)	r^{-2}	$\ln r$ (7.6)	r^{-1}
Loop	r^{-2} (3.30)	r^{-3}	General: r^{-2} (5.33) Symmetric: r^{-3} (5.33)	r^{-3} r^{-4}

For infinitesimal symmetric disclination loops both u_n^T and e_{mn} vanish. A more accurate calculation of the asymptotic displacement from a small but finite disclination loop can also be made from (5.33) by expanding the Green's tensor as a Taylor series in $\mathbf{r}'$ for a few terms and integrating over S. The details will be worked out in a subsequent publication [26], where it will be shown that for the isotropic case the results reduce to those of Li and Gilman [20].

We now find the total distortion from (5.33)

$$\begin{aligned}
u^T_{n,m}(\mathbf{r}) &= \int_S C_{ijkl}G_{jn,im}(\mathbf{R})\{b_l+\epsilon_{lqr}\Omega_q(x'_r-x^\circ_r)\}dS'_k \\
&= \oint_L \epsilon_{pmk}C_{ijkl}G_{jn,i}(\mathbf{R})\{b_l+\epsilon_{lqr}\Omega_q(x'_r-x^\circ_r)\}dL'_p \\
&\quad+\int_S C_{ijkl}G_{jn,i}(\mathbf{R})\epsilon_{lqr}\Omega_q\delta_{rm}dS'_k \\
&\quad+\int_S C_{ijkl}G_{jn,ik}(\mathbf{R})\{b_l+\epsilon_{lqr}\Omega_q(x'_r-x^\circ_r)\}dS'_m \\
&= \oint_L \epsilon_{pmk}C_{ijkl}G_{jn,i}(\mathbf{R})\{b_l+\epsilon_{lqr}\Omega_q(x'_r-x^\circ_r)\}dL'_p \\
&\quad+\int_S \epsilon_{pmk}C_{ijkl}G_{jn,i}(\mathbf{R})\Omega_p dS'_l+\beta^*_{mn}(\mathbf{r}). \qquad (5.34)
\end{aligned}$$

Here the second equality follows from Stokes' theorem (A4), and the third from (2.7), (B5), and (5.12). Now if we call $\beta_{mn}=u^T_{n,m}-\beta^*_{mn}$ the "elastic distortion," it is not a state quantity because the second integral in (5.34) cannot be written as a line integral. The proof of this consists merely in showing that the integrand in this term is not divergence-free; using (2.7) we find

$$\begin{aligned}[\epsilon_{pmk}C_{ijkl}G_{jn,i}(r)]_{,l} &= \epsilon_{pmk}C_{ijkl}G_{jn,il}(\mathbf{r})\\ &= -\epsilon_{pmn}\delta(r)\\ &\neq 0. \end{aligned} \tag{5.35}$$

If we compare the second integral of (5.34) with (3.30) we notice a great deal of similarity. As is well-known the jump of u_n^T in (3.30) across the surface S is given by $[u_n^T]=b_n$ according to (3.18). From this we deduce that the second integral in (5.34) leads to a jump of the total distortion β_{mn}^T across the surface S of $[\beta_{mn}^T]=\epsilon_{pmn}\Omega_p$. This is a jump in rotation of $[\omega_p^T]=1/2\epsilon_{pmn}[\beta_{mn}^T]=\Omega_p$, in agreement with (5.6). In the older approach to dislocation theory where the surface S was ignored, the displacement of a discrete dislocation line was regarded as a multiple-valued function with a period of the Burgers vector. Equation (5.34) shows that from this point of view the rotation (or distortion) of a discrete disclination line is a multiple-valued function with a period of the Frank vector.

The elastic strain is from (4.14), (5.14), and (5.34)

$$e_{mn}(\mathbf{r})=\oint_L \epsilon_{pmk}C_{ijkl}G_{jn,i}(\mathbf{R})\{b_l+\epsilon_{lqr}\Omega_q(x'_r-x_r^0)\}dL'_{p(mn)}+\int_S \epsilon_{pmk}C_{ijkl}G_{jn,i}(\mathbf{R})\Omega_p dS'_{l(mn)}. \tag{5.36}$$

The integrand of the second integral now *is* divergence-free and so it can be written as a line integral

$$-\int_S \epsilon_{qsl}I_{mnpq,s}(\mathbf{R})\Omega_p dS'_l=-\oint_L I_{mnpq}(\mathbf{R})\Omega_p dL'_q$$

by (4.28) and Stokes' theorem. Hence

$$e_{mn}(\mathbf{r})=\oint_L \epsilon_{pmk}C_{ijkl}G_{jn,i}(\mathbf{R})\{b_l+\epsilon_{lqr}\Omega_q(x'_r-x_r^0)\}dL'_{p(mn)}-\oint_L I_{mnpq}(\mathbf{R})\Omega_p dL'_p. \tag{5.37}$$

This result could of course also be obtained directly from (4.29), (5.19–20) and (B11). This is the relation we sought. We see that the elastic strain can be written as a line integral along the discrete defect line. Therefore it is a state quantity.

Equation (5.37) also allows us to estimate the asymptotic behavior of the strain at large distances from a straight disclination line. The second integral will give the dominant term. The incompatibility source tensor $I_{mnpq}(\mathbf{r})$ varies as r^{-1}. Due to the integration then $e_{mn}(\mathbf{r})$ will vary as $\ln r$ as $r\to\infty$. Since the displacement is an integral of the strain, it will vary as $r \ln r$ as $r\to\infty$. These results are also listed in table 3.

We wish next to derive the elastic bend-twist. First we find the derivative of the total distortion (5.34)

$$u_{n,ms}^T(\mathbf{r})=\oint_L \epsilon_{pmk}C_{ijkl}G_{jn,is}(\mathbf{R})\{b_l+\epsilon_{lqr}\Omega_q(x'_r-x_r^0)\}dL'_p+\int_S \epsilon_{pmk}C_{ijkl}G_{jn,is}(\mathbf{R})\Omega_p dS'_l+\beta^*_{mn,s}(\mathbf{r}). \tag{5.38}$$

The second term becomes by Stokes' theorem

$$\oint_L \epsilon_{pmk}\epsilon_{qsl}C_{ijkl}G_{jn,i}(\mathbf{R})\Omega_p dL'_q+\int_S \epsilon_{pmk}C_{ijkl}G_{jn,il}(\mathbf{R})\Omega_p dS'_s.$$

The second term in this expression becomes by (2.7), (B5), and (5.13)

$$-\int_S \epsilon_{pmk}\delta_{kn}\delta(\mathbf{R})\Omega_p dS'_s = -\epsilon_{pmn}\delta_s(S)\Omega_p = \epsilon_{pmn}\phi^*_{sp}(\mathbf{r}).$$

Hence (5.38) becomes

$$\begin{aligned} u^T_{n,\,ms}(\mathbf{r}) = &\oint_L \epsilon_{pmk}C_{ijkl}G_{jn,\,is}(\mathbf{R})\{b_l + \epsilon_{lqr}\Omega_q(x'_r - x^0_r)\}dL'_p \\ &+ \oint_L \epsilon_{pmk}\epsilon_{qsl}C_{ijkl}G_{jn,\,i}(\mathbf{R})\Omega_p dL'_q \\ &+ \beta^*_{mn,\,s}(\mathbf{r}) + \epsilon_{pmn}\phi^*_{sp}(\mathbf{r}). \end{aligned} \tag{5.39}$$

We now find from (4.15) and (5.15)

$$\begin{aligned} \kappa_{st}(\mathbf{r}) = 1/2 &\oint_L \epsilon_{tmn}\epsilon_{pmk}C_{ijkl}G_{jn,\,is}(\mathbf{R})\{b_l + \epsilon_{lqr}\Omega_q(x'_r - x^0_r)\}dL'_p \\ + 1/2 &\oint_L \epsilon_{tmn}\epsilon_{pmk}\epsilon_{qsl}C_{ijkl}G_{jn,\,i}(\mathbf{R})\Omega_p dL'_q. \end{aligned} \tag{5.40}$$

This could also have been obtained directly by substituting into (4.34) from (5.19–20). It is also seen to be a state quantity.

This section has extended the results for a discrete dislocation line of section 3.2.2 to a more general discrete defect line. The central result there, equation (3.32), has been replaced by (5.37) and (5.40). Again we see the elastic distortion of dislocation theory is generalized into the basic elastic fields of defect theory.

6. Continuous Distribution of Infinitesimal Defect Loops

In section 4.1 the defects were defined by (4.1–2) and the characteristic vectors by (4.5–6) in terms of the given basic plastic fields, strain and bend-twist. Eliminating the basic plastic fields from these definitions led to (4.7–8), relations between the characteristic vectors and the defect densities. These equations could alternatively be used to define the defect densities in terms of the characteristic vectors, if they are prescribed in a suitable manner. It is convenient to put these relations into differential form. First we define the disclination density. When the defects are continuously distributed the disclination density tensor is defined locally by

$$\theta_{pq} \equiv \frac{\Delta\Omega_q}{\Delta S_p}. \tag{6.1}$$

For a distribution of discrete defect lines this represents the average disclination density where $\Delta\Omega_q$ is the qth component of the resulting Frank vectors of all the disclinations which pierce through a surface element ΔS_p oriented normal to the x_p direction at the given point. In view of (4.7) it is necessary to modify the definition (3.33) for the dislocation density α_{pl}. For an inhomogeneous continuous distribution of defects we define it locally by the relation

$$\alpha_{pl} - \epsilon_{lqr}\theta_{pq}x_r \equiv \frac{\Delta B_l}{\Delta S_p}. \tag{6.2}$$

For discrete defect lines this relation will give us the average dislocation density, where ΔB_l is the lth component of the resulting total Burgers vectors of all the defects (dislocations and disclinations) which pierce through a surface element ΔS_p normal to the x_p direction at the given point.

Reference to (5.31) suggests that a more convenient way to write (6.2) is

$$\alpha_{pl} \equiv \frac{\Delta[u_l]}{\Delta S_p}, \tag{6.3}$$

where we have defined

$$\Delta[u_l] \equiv \Delta B_l + \epsilon_{lqr} \Delta\Omega_q x_r, \tag{6.4}$$

as suggested by (5.1). For discrete defect lines $\Delta[u_l]$ represents the lth component of the resulting displacement jumps of all the defect lines which pierce through a surface element ΔS_p oriented normal to the x_p direction at the given point. Relation (6.3) is what would be expected if the dislocation density is determined by the local plastic deformation, regardless of whether it is due to discrete dislocations or disclinations. For generality relation (3.33) could also have have been put in the form (6.3) by reference to (3.18).

We next consider a continuous distribution of infinitesimal defect loops differently oriented in space. We introduce the disclination loop density tensor following Kroupa's [5] line of reasoning. We can define it as follows: ζ_{kq} represents the flux of disclination (or Frank vector) in the x_q direction that encloses a unit vector in the x_k direction. When the loops are continuously and inhomogeneously distributed the density tensor ζ_{kq} is a function of the position and is defined locally by

$$\zeta_{kq} \equiv \frac{\Delta\Omega_q}{\Delta L_k}. \tag{6.5}$$

For a distribution of discrete loops this represents the average disclination loop density where now $\Delta\Omega_q$ is the qth component of the resulting Frank vectors of all the loops which are pierced by the line element ΔL_k oriented in the x_k direction at the given point. It is now also necessary to modify the definition (3-34) for the dislocation loop density. For a continuous distribution of defect loops it is defined locally by

$$\gamma_{kl} \equiv \frac{\Delta[u_l]}{\Delta L_k}. \tag{6.6}$$

For discrete loops this represents the average dislocation loop density, where now $\Delta[u_l]$ is the lth component of the resulting displacement jumps of all the loops which are pierced by the line element ΔL_k oriented in the x_k direction at the given point. By (6.4) we also have the alternative definition

$$\gamma_{kl} \equiv \frac{\Delta B_l}{\Delta L_k} + \epsilon_{lqr} \frac{\Delta\Omega_q}{\Delta L_k} x_r, \tag{6.7}$$

where for discrete loops ΔB_l is the lth component of the resulting total Burgers vectors of all the defect loops which are pierced by the line element ΔL_k oriented in the x_k direction at the given point.

To derive the relations between the defect loop densities and the basic plastic fields, we first combine (6.5) and (6.7) into

$$\gamma_{kl} - \epsilon_{lqr} \zeta_{kq} x_r = \frac{\Delta B_l}{\Delta L_k}. \tag{6.8}$$

The relations (6.8) and (6.5) are easily converted to integral form

$$B_l = \oint_\lambda (\gamma_{kl} - \epsilon_{lqr} \zeta_{kq} x_r)\, dL_k, \tag{6.9}$$

$$\Omega_q = \oint_\lambda \zeta_{kq}\, dL_k. \tag{6.10}$$

Now we compare these relations with the definitions (4.5–6). The integrands can only differ by a gradient with respect to x_k. Therefore we can set

$$\kappa^P_{kq} = -\zeta_{kq} + \omega^P_{q,k}, \tag{6.11}$$

$$e^P_{kl} - \epsilon_{lqr}\kappa^P_{kq}x_r = -\gamma_{kl} + \epsilon_{lqr}\zeta_{kq}x_r + (u^P_l - \epsilon_{lqr}\omega^P_q x_r)_{,k}, \tag{6.12}$$

where ω^P_q and u^P_l are arbitrary vector fields, subject only to the condition that e^P_{kl} is symmetric. This last condition will provide a relation between them, as we shall show. If we substitute (6.11) into (6.12) we obtain

$$e^P_{kl} = -\gamma_{kl} - \epsilon_{klq}\omega^P_q + u^P_{l,k}. \tag{6.13}$$

Now the symmetry condition on e^P_{kl} gives the relation between ω^P_q and u^P_l

$$\omega^P_q = -1/2\epsilon_{klq}(\gamma_{kl} - u^P_{l,k}). \tag{6.14}$$

From this we get for (6.13) and (6.11)

$$e^P_{kl} = -\gamma_{(kl)} + u^P_{(l,k)}, \tag{6.15}$$

$$\kappa^P_{mq} = -\zeta_{mq} - 1/2\epsilon_{klq}\gamma_{kl,m} + 1/2\epsilon_{klq}u^P_{l,km}. \tag{6.16}$$

These are the relations that identify the basic plastic fields for a continuous distribution of defect loops. With them we can find all the relations derived in section 4 in terms of a loop distribution. For example, from (4.1–2) we find

$$\alpha_{pl} = \epsilon_{pmk}(\gamma_{kl,m} + \epsilon_{klq}\zeta_{mq}), \tag{6.17}$$

$$\theta_{pq} = \epsilon_{pmk}\zeta_{kq,m}. \tag{6.18}$$

This is the fundamental relationship between the defect loop densities and the corresponding defect densities. We see from (6.17) that the disclination loop density ζ_{mq} also contributes to the dislocation density, in addition to the dislocation loop density γ_{kl}.

Since a plastic displacement such as u^P_l in (6.15–16) does not contribute to the elastic fields (c.f. section 2.6), we can set $u^P_l = 0$ without loss of generality, and so we can use

$$e^P_{kl} = -\gamma_{(kl)}, \tag{6.19}$$

$$\kappa^P_{mq} = -\zeta_{mq} - 1/2\epsilon_{klq}\gamma_{kl,m} \tag{6.20}$$

for the purpose of calculating the fields of a continuous distribution of infinitesimal defect loops. Furthermore, we note from (2.15) and (6.19) that only the dislocation loop density γ_{kl} will contribute to the total displacement. Hence the elastic strain and stress are unaffected by the disclination loop density, ζ_{mq}, as we already hinted at in section 5.3. We also note that all the above results reduce to those of section 3.3 when the disclination loop density vanishes, $\zeta_{mq} = 0$.

Mura [4] generalized his "plastic distortion" and "plastic rotation," which he had defined for a discrete loop as in section 5.2, to a continuous distribution. We shall now interpret his approach. If we compare (5.14–15) with (6.19–20) we can make the following identification:

$$\beta^*_{(kl)} = -\gamma_{(kl)}, \tag{6.21}$$

$$\phi^*_{mq} + \frac{1}{2}\epsilon_{klq}\beta^*_{kl,m} = -\zeta_{mq} - \epsilon_{klq}\gamma_{kl,m}. \tag{6.22}$$

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These equations can be solved for Mura's plastic quantities as follows:

$$\beta^*_{kl} = -\gamma_{kl} - \epsilon_{klq}\omega^*_q, \tag{6.23}$$

$$\phi^*_{mq} = -\zeta_{mq} + \omega^*_{q,m}, \tag{6.24}$$

where ω^*_q is an arbitrary vector field. These relations identify Mura's quantities for a continuous distribution of defect loops. The basic plastic fields are obtained in terms of Mura's quantities by substituting in (6.19–20)

$$e^P_{kl} = \beta^*_{(kl)}, \tag{6.25}$$

$$\kappa^P_{mq} = \phi^*_{mq} + \frac{1}{2}\epsilon_{klq}\beta^*_{kl,m}. \tag{6.26}$$

These relations are identical in form with (5.14–15), but are now also valid for a continuous distribution. We find the defect densities by substituting in (4.1–2)

$$\alpha_{pl} = -\epsilon_{pmk}(\beta^*_{kl,m} + \epsilon_{klq}\phi^*_{mq}), \tag{6.27}$$

$$\theta_{pq} = -\epsilon_{pmk}\phi^*_{kq,m}, \tag{6.28}$$

which are identical to relations in section 5.2. The interesting vector ω^*_q does not affect the basic plastic fields and the defect densities, and hence it does not affect the elastic fields either. In (6.23) it contributes the antisymmetric term $-\epsilon_{klq}\omega^*_q$ to β^*_{kl}. Mura [30] has called the dislocations resulting from an antisymmetric plastic distortion an *impotent* distribution of dislocations, because, as can be deduced from (2.15), such a distortion does not contribute to the total displacement, and hence gives no elastic fields. In general an antisymmetric plastic distortion will give a finite dislocation density, c.f. (3.4). However, the term $\omega^*_{q,m}$ in (6.24) is exactly right to annihilate both defect densities (6.27–28) due to ω^*_q. Hence, we can set $\omega^*_q = 0$ without loss of generality for the purpose of calculating the elastic fields due to a given distribution of β^*_{kl} and ϕ^*_{kq}, or

$$\beta^*_{kl} = -\gamma_{kl}, \tag{6.29}$$

$$\phi^*_{mq} = -\zeta_{mq}. \tag{6.30}$$

This shows that Mura's "plastic distortion" and "plastic rotation," introduced in section 5 for a discrete defect line, can be interpreted as the dislocation and disclination loop densities, except for a minus sign. This then resolves a difference we had with Mura. Relations (6.25–26) are the basic relations that connect Mura's approach with ours. For example the characteristic vectors are found from (4.5–6) to be

$$B_l = -\oint_\lambda (\beta^*_{kl} - \epsilon_{klq}\phi^*_{kq}x_r)\,dL_k, \tag{6.31}$$

$$\Omega_q = -\oint_\lambda \phi^*_{kq}\,dL_k, \tag{6.32}$$

which correspond to (6.9–10).

We can now also give an interpretation to (5.12–13) in terms of infinitesimal loops. To construct a discrete defect line L, we first distribute a constant density of infinitesimal disclination loops of strength Ω_q over any surface S whose boundary is L. This distribution only gives a rotation across S, but no stress, and furthermore no unique axis is defined. Rather, each infinitesimal loop has its own axis, so that the resultant axis is continuously distributed over S. Now we add a distribution of infinitesimal dislocation loops to S, consisting of two parts. One, containing the dislocation Burgers vector, is a constant distribution of strength b_l, which gives the discrete dislocation line. The other, containing the Frank vector Ω_q, has just the right strength to bring the axis of each infinitesimal disclination loop to the point x_r^0. We see that this is done by increasing the dislocation loop strength proportional to the distance from the chosen axis. The combination of this linear infinitesimal dislocation loop density with the constant infinitesimal disclination loop density mentioned above gives the discrete disclination line. It is the infinitesimal dislocation loop distribution that gives rise to the elastic strain of the discrete disclination line.

So we conclude that for a finite defect loop the defect loop densities are given by

$$\gamma_{kl}(\mathbf{r}) = \delta_k(S)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\}, \tag{6.33}$$

$$\zeta_{kq}(\mathbf{r}) = \delta_k(S)\Omega_q. \tag{6.34}$$

The vector $\delta_k(S)$ is the Dirac delta function on the surface S and it is always normal to S. In a plane we have a twist or a wedge disclination loop according to whether the Frank vector is normal or parallel to S, respectively. Therefore (6.34) shows that the diagonal and off-diagonal components of ζ_{kq} represent the twist and wedge components of the disclination loop density, respectively (see table 1). Equations (6.33–34) show how to make the transition from a continuous distribution of infinitesimal defect loops to a finite defect loop. By (B15) and (5.17) we see that they satisfy (6.9–10), which also remain valid for many finite loops, and therefore can be used for the transition from finite loops to a continuous distribution: For many finite defect loops the average dislocation and disclination loop densities γ_{kl} and ζ_{kq} represent the sum of the x_l component of the displacement jump vectors and the x_q component of the Frank vectors, respectively, of all the loops whose surfaces are intersected by a unit vector in the x_k direction.

The infinitesimal defect loop density tensors ζ_{kq} and γ_{kl} were defined in this section independent of the definitions of the defects densities in section 4.1 and the discrete defects in section 5.1. What is the correlation?

Let us first examine the relation between continuous distributions of defects and loop densities. For dislocations only there is a straightforward relation between them given by (3.37). When disclinations are introduced there is a similar straightforward relation between the disclination loop density and the disclination density tensor, given by (6.18). However, the dislocation density now contains a contribution from the disclination loops, equation (6.17). This means that the two definitions of the "dislocation" are not identical, and that a certain amount of mixing occurs in going from one formulation to the other.

Now let us examine the relation between a discrete defect line and a continuous distribution of loops. For dislocations only there is again a straightforward relation between the dislocation loop density and the Burgers vector, given by (3.39), or between the Burgers vector and the dislocation loop density, given by (3.35). When disclinations are introduced there is a similar straightforward relation between the disclination loop density and the Frank vector, given by (6.34), or between the Frank vector and the disclination loop density, given by (6.10). However, the dislocation loop density for a discrete defect line now contains a contribution from the Frank vector, equation (6.33), or the general Burgers vector contains a contribution from the disclination loop density, equation (6.9). Therefore the definitions of the dislocation are not identical in both approaches, and as we see from the relations quoted, a certain amount of mixing occurs in going from one formulation to the other.

There are, therefore, at least three independent ways to define the dislocation content of defects: in terms of a continuous distribution, a discrete line, or a continuous distribution of infinitesimal loops. For dislocations only, these definitions are equivalent but with disclinations they are essentially different.

7. The Discrete Dipole Line

7.1. Basic Relations: The Biaxial Dipole

Extending Kroupa's [18] definition of the dislocation dipole, we define the discrete dipole line as a close pair of discrete defect lines with opposite characteristic vectors. We call it the biaxial dipole because this defect would have two rotation axes. We wish to present in this section some of the relations analogous to those for the dislocation dipole in section 3.4.2. From (3.42), we find the defect densities of the biaxial dipole conjugate to the basic dislocation and disclination densities and θ_{pq} to be

$$\alpha^D_{pl}(\mathbf{r}) = -\xi_m \alpha_{pl,\,m}(\mathbf{r}), \tag{7.1}$$

$$\theta^D_{pq}(\mathbf{r}) = -\xi_m \theta_{pq,\,m}(\mathbf{r}). \tag{7.2}$$

By (5.19–20) these relations become for the discrete dipole line

$$\alpha^D_{pl}(\mathbf{r}) = -\delta_{p,\,m}(L)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x^\circ_r)\}\xi_m - \delta_p(L)\epsilon_{lqr}\Omega_q\xi_r, \tag{7.3}$$

$$\theta^D_{pq}(\mathbf{r}) = -\delta_{p,\,m}(L)\Omega_q\xi_m. \tag{7.4}$$

The meaning of the displacement ξ_m is as follows: We let $\xi_m \to 0$, $b_l \to \infty$, and $\Omega_q \to \infty$ in such a way that $b_l\xi_m$ and $\Omega_q\xi_m$ remain constant.

The displacement of the dipole line is from (3.44)

$$u^D_n(\mathbf{r}) = -\xi_m u^T_{n,\,m}(\mathbf{r}) \tag{7.5}$$

$$\begin{aligned} &= -\oint_L \epsilon_{pmk} C_{ijkl} G_{jn,\,i}(\mathbf{R})\{b_l + \epsilon_{lqr}\Omega_q(x'_r - x^0_r)\}\xi_m dL'_p \\ &\quad -\int_S \epsilon_{pmk} C_{ijkl} G_{jn,\,i}(\mathbf{R})\Omega_p\xi_m dS'_l \\ &\quad + \delta_m(S)\{b_l + \epsilon_{nqr}\Omega_q(x_r - x^0_r)\}\xi_m, \end{aligned} \tag{7.6}$$

where we have used (5.34) and (5.12). In this expression the third line can be made to vanish by choosing S near the line to lie along ξ, or $\xi_m dS'_m = 0$ along L. Nevertheless, due to the surface integral, the displacement is not a state quantity in contrast to the case of the dislocation dipole.

From (7.1–6) all other relevant quantities for a discrete biaxial dipole line can be derived if desired.

Equation (7.6) allows us to estimate the asymptotic behavior of the displacement at large distances from a straight biaxial dipole line. It is easily deduced that it will vary as $\ln r$ as $r \to \infty$. Hence the strain will vary as r^{-1} as $r \to \infty$. These results are also listed in table 3.

7.2. Influence of the Axis: The Uniaxial Dipole

The dipole of section 7.1 is obtained from the basic discrete defect line of section 5 by translating it through the infinitesimal distance $\boldsymbol{\xi}$, including the axis of rotation, and then subtracting the basic defect at its original position.

In this section we first wish to isolate the influence of the motion of the axis, when the position of the defect line is held fixed. We therefore consider the following defect, again composed of two parts: the first is obtained from the basic defect by translating its axis only through an infinitesimal distance ξ, and the second part is the negative of the basic defect at its original position. The defect densities of this defect are given by

$$\alpha^A_{pl} = \xi_m \partial \alpha_{pl} / \partial x^0_m \,, \tag{7.7}$$

$$\theta^A_{pq} = \xi_m \partial \theta_{pq} / \partial x^0_m . \tag{7.8}$$

By (5.19–20) these relations become

$$\alpha^A_{pl}(\mathbf{r}) = -\delta_p(L) \epsilon_{lqr} \Omega_q \xi_r , \tag{7.9}$$

$$\theta^A_{pq}(\mathbf{r}) = 0. \tag{7.10}$$

The displacement is

$$u^A_n(\mathbf{r}) = \xi_m \partial u^T_n(\mathbf{r}) / \partial x^0_m \tag{7.11}$$

$$= -\int_S C_{ijkl} G_{jn,\,i}(\mathbf{R}) \epsilon_{lqr} \Omega_q \xi_r dS'_k$$

$$= -\int_S \epsilon_{pmk} C_{ijkl} G_{jn,\,i}(\mathbf{R}) \Omega_p \xi_m dS'_l , \tag{7.12}$$

where the second equality follows from (5.33), and the third by rearranging the indices using the symmetry condition (2.3).

On comparing (7.9) and (7.12) with (3.23) and (3.30), we see that they represent the dislocation density and displacement of a discrete dislocation line with Burgers vector

$$b_l = -\epsilon_{lqr} \Omega_q \xi_r . \tag{7.13}$$

This was to be expected, since the motion of the axis through the distance ξ_r has the effect of translating the two sides of S by the distance b_l given in (7.13), as we can see from (5.5). This conclusion complements the statement in section 5.2 that the axis of a discrete defect line can be moved by adding a dislocation to the line, c.f. (5.23).

Next we want to examine the effect of holding the axis fixed and moving the defect line only. This type of the defect will be called a uniaxial dipole because it has only one axis. It could alternatively have been used as the definition of the discrete dipole line. It is composed of the following two parts: the first is obtained from the basic discrete defect line by translating it through an infinitesimal distance ξ keeping its axis fixed, and the second part is the negative of the basic defect at its original position. The resulting fields are simply the difference between those in section 7.1 and the above. So the dislocation density and displacement of the uniaxial dipole are

$$\alpha^L_{pl}(\mathbf{r}) = \alpha^D_{pl}(\mathbf{r}) - \alpha^A_{pl}(\mathbf{r}) \tag{7.14}$$

$$= -\delta_{p,m}(L) \{ b_l + \epsilon_{lqr} \Omega_q (x_r - x^0_r) \} \xi_m , \tag{7.15}$$

$$u^L_n(\mathbf{r}) = u^D_n(\mathbf{r}) - u^A_n(\mathbf{r}) \tag{7.16}$$

$$= -\oint_L \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R}) \{ b_l + \epsilon_{lqr} \Omega_q (x'_r - x^0_r) \} \xi_m dL'_p , \tag{7.17}$$

whereas the disclination density is the same for either type of dipole,

$$\theta^L_{pq} = \theta^D_{pq} . \tag{7.18}$$

We note that the displacement of the uniaxial dipole is a state quantity.

A special case of (7.17) is of particular interest, namely, the one corresponding to the wedge disclination coinciding with its axis as the basic defect. In this case $b_l=0$. For a wedge disclination the Frank vector is parallel to the disclination line, $\mathbf{\Omega} \| d\mathbf{L}'$. The point $\mathbf{r}'$ lies on the line and the point $\mathbf{r}^0$ on the axis. Since the line coincides with the axis, the difference, is also parallel to the line, $(\mathbf{r}'-\mathbf{r}^0) \| d\mathbf{L}'$. So $\Omega \| (\mathbf{r}'-\mathbf{r}^0)$, and hence $\epsilon_{lqr}\Omega_q(x_r'-x_r^0)=0$. We conclude that (7.17) vanishes for this case. Therefore $u_n^D=u_n^A$. This means that the conjugate dipole, corresponding to this wedge disclination with Frank vector Ω_q, is the dislocation with Burgers vector b_l given by (7.13). It is an edge dislocation because $\mathbf{b} \perp \mathbf{\Omega} \| d\mathbf{L}'$. Eshelby [21] used this approach to give a simple derivation of the elastic field of an edge dislocation, when the field of the wedge disclination is known.

Equation (7.17) shows that the asymptotic behavior of the displacement at large distances from a straight uniaxial dipole is the same as for a biaxial dipole. Table 3 shows there is a gap in the asymptotic behavior between a disclination dipole and loop, i.e., there is no disclination type defect with the r^{-1} behavior for the displacement.

8. Application to Dislocations

The purpose of this section is to check the internal consistency of the results for dislocations and the more general defects. It contains no new material. We want to show that the results of section 3 fall out of section 4–6, when no disclinations are present, i.e., when the plastic deformation of the body is completely described by the plastic distortion β_{kl}^P.

8.1. Continuous Distribution of Dislocations

8.1.1. Geometry

The plastic strain and rotation are now given by (c.f. 3.2–3)

$$e_{kl}^P=\beta_{(kl)}^P, \tag{8.1}$$

$$\omega_q^P=1/2\epsilon_{klq}\beta_{kl}^P, \tag{8.2}$$

so that

$$e_{kl}^P+\epsilon_{klq}\omega_q^P=\beta_{kl}^P, \tag{8.3}$$

in agreement with (3.1). The plastic bend-twist is (c.f. 3.12)

$$\kappa_{kq}^P=\omega_{q,k}^P. \tag{8.4}$$

Hence we find from (4.1)

$$\alpha_{pl}=-\epsilon_{pmk}(e_{kl,m}^P+\epsilon_{klq}\omega_{q,m}^P)=-\epsilon_{pmk}\beta_{kl,m}^P, \tag{8.5}$$

in agreement with (3.4), and from (4.2)

$$\theta_{pq}=-\epsilon_{pmk}\omega_{q,km}^P=0, \tag{8.6}$$

as expected. The Burgers vector is found from (4.5)

$$\begin{aligned} B_l&=-\oint_\lambda (e_{kl}^P-\epsilon_{lqr}\omega_{q,k}^P x_r)dL_k \\ &=-\oint_\lambda (e_{kl}^P+\epsilon_{lqk}\omega_q^P)dL_k \\ &=-\oint_\lambda \beta_{kl}^P dL_k. \end{aligned} \tag{8.7}$$

Here the first equality follows from (8.4), the second from a partial integration where the integrated part vanishes around the closed curve λ, and the third from (8.3). The result agrees with (3.7). Furthermore we also see that (8.6) in (4.7) agrees with (3.8). The Frank vector is found from (4.6) and (8.4):

$$\Omega_q = -\oint_\lambda \omega_{q,k} dL_k = 0, \tag{8.8}$$

as expected. This also agrees with (8.6) in (4.8).

In a similar way it is easily shown that (4.17) and (4.20) reduce to (3.10) and (3.11), whereas (4.18) and (4.21) vanish.

8.1.2. Statics

The κ_{lp}^P term in (4.23) becomes by (8.4)

$$-\int \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R}) \omega_{p,l'}^P(\mathbf{r}') dV' = -\int \epsilon_{pmk} C_{ijkl} G_{jn,il}(\mathbf{R}) \omega_p^P(\mathbf{r}') dV'$$

$$= \epsilon_{pmn} \omega_p^P(\mathbf{r}).$$

Here the first equality follows by a partial integration, and the second by (2.7). Hence we have for the total distortion (4.23).

$$u_{n,m}^T(\mathbf{r}) = \int \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R}) \alpha_{pl}(\mathbf{r}') dV' + e_{mn}^P(\mathbf{r}) + \epsilon_{pmn} \omega_p^P(\mathbf{r}), \tag{8.9}$$

in agreement with (3.16) and (3.1). From (4.14) we find the elastic strain

$$e_{mn}(\mathbf{r}) = \int \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R}) \alpha_{pl}(\mathbf{r}') dV'_{(mn)}. \tag{8.10}$$

This relation can also be obtained from (4.29) with (8.6). It is in agreement with (3.13) and (3.17).

We next find the bend-twist. From (8.9) and (8.4) we find

$$u_{n,ms}^T(\mathbf{r}) = \int \epsilon_{pmk} C_{ijkl} G_{jn,is}(\mathbf{R}) \alpha_{pl}(\mathbf{r}') dV' + e_{mn,s}^P(\mathbf{r}) + \epsilon_{pmn} \kappa_{sp}^P(\mathbf{r}). \tag{8.11}$$

Hence we find from (4.15)

$$\kappa_{st}(\mathbf{r}) = 1/2 \int \epsilon_{tmn} \epsilon_{pmk} C_{ijkl} G_{jn,is}(\mathbf{R}) \alpha_{pl}(\mathbf{r}') dV'. \tag{8.12}$$

This relation can also be obtained from (4.34) with (8.6). We see it is in agreement with (3.14) and (3.17).

8.2. The Discrete Dislocation Line

8.2.1. Geometry

When a discrete defect line contains no disclination, $\Omega_q=0$, as we saw from (8.8). For this case (5.5) reduces to (3.18) and (5.6) becomes

$$[\omega_q(\mathbf{r})]=0, \tag{8.13}$$

while (5.7) reduces to (3.19) and (5.8) reduces to (3.20).

Next we find that (5.12–13) become

$$\beta^*_{kl}(\mathbf{r})=-\delta_k(S)b_l, \tag{8.14}$$

$$\phi^*_{kq}(\mathbf{r})=0. \tag{8.15}$$

This shows from (3.21) that

$$\beta^*_{kl}=\beta^P_{kl}, \tag{8.16}$$

as expected, since both sides represent the dislocation loop density by (6.29) and (3.38). Equation (8.15) shows that the disclination loop density vanishes. Now (5.14–15) become from (8.14–15)

$$e^P_{kl}(\mathbf{r})=-1/2[\delta_k(S)b_l+\delta_l(S)b_k], \tag{8.17}$$

$$\left.\begin{aligned}\kappa^P_{mq}(\mathbf{r})&=-1/2\epsilon_{klq}\delta_{k,m}(S)b_l,\\ (\kappa^P_{kq}(\mathbf{r})&=-1/2\epsilon_{lqr}\delta_{r,k}(S)b_l).\end{aligned}\right. \tag{8.18}$$

We see that (8.17) is in agreement with (3.2) and (3.21) and that (8.18) agrees with (3.12) and (3.21).

From (5.17) we find that

$$B_l=b_l, \tag{8.19}$$

showing that the total Burgers vector reduces to the dislocation Burgers vector. From (4.1) and (8.17–18) we find

$$\alpha_{pl}(\mathbf{r})=\epsilon_{pmk}\delta_{k,m}(S)b_l=\delta_p(L)b_l. \tag{8.20}$$

This result can also be obtained directly from (5.19) and is in agreement with (3.23). Next we find from (4.2) and (8.18) or directly from (5.20)

$$\theta_{pq}(\mathbf{r})=1/2\,\epsilon_{pmk}\,\epsilon_{lqr}\,\delta_{r,km}\,(S)\,b_l=0, \tag{8.21}$$

as expected, in agreement with (8.6).

8.2.2. Statics

Again for $\Omega_q=0$, we see that (5.33) reduces to (3.30), and (5.34) reduces to (3.31) with (8.16). The elastic strain is

$$e_{mn}(\mathbf{r})=\oint_L \epsilon_{pmk}C_{ijkl}G_{jn,i}(\mathbf{R})b_l dL'_{P(mn)}. \tag{8.22}$$

This result can be found in two ways: first from (5.37), or second from (4.29) with (8.20–21). It is in agreement with (3.13) and (3.32).

Next we find the elastic bend-twist from (5.40)

$$\kappa_{st}(\mathbf{r}) = 1/2 \oint_L \epsilon_{tmn}\epsilon_{pmk} C_{ijkl} G_{jn,\, is}(\mathbf{R}) b_l dL'_p. \tag{8.23}$$

This could also have been found from (4.34) with (8.20–21), and is in agreement with (3.14) and (3.32).

So we have shown in this section that the more general defect theory, including disclinations, completely reduces to the well-known dislocation theory in the special case that the disclinations vanish.

9. The "Dislocation Model" of a Discrete Defect Line

Li and Gilman [20] considered the finite disclination loop as a continuous distribution of dislocations, and called this the "dislocation model" of the disclination. Mura [22] used the same concept, which he ascribed to Eshelby, discussing also the case where he replaced a wedge disclination by a semi-infinite edge dislocation wall. We wish to make clear the distinction between the two concepts.

We start with the observation, made in section 6, that only the dislocation loop density, or Mura's "plastic distortion" β^*_{kl}, contributes to the elastic strain. For a discrete defect line, this quantity is given by (5.12). So the elastic strain obtained from this expression does not depend on what we choose for the disclination loop density, or Mura's "plastic rotation" ϕ^*_{kq}. If we choose it to vanish

$$\phi^*_{kq} = 0, \tag{9.1}$$

then we have a distribution of dislocation loops over the surface S that gives exactly the same elastic strain as the discrete defect line of section 5. We call the corresponding dislocation distribution the *dislocation model* of the defect line. We note that this dislocation model is clearly a different defect from the discrete defect line it corresponds to. Specifically, we obtain the dislocation model by setting

$$\beta^P_{kl} = \beta^*_{kl}, \tag{9.2}$$

where

$$\beta^*_{kl}(\mathbf{r}) = -\,\delta_k(S)\{b_l + \epsilon_{lqr}\Omega_q(x_r - X^\circ_r)\}\,, \tag{9.3}$$

and where b_l and Ω_q are constants. We can then use the methods of section 3.1 to find any other desired relations. Furthermore (5.14–16) become by (9.1)

$$e^P_{kl} = \beta^*_{(kl)}, \tag{9.4}$$

$$\kappa^P_{mq} = 1/2\epsilon_{klq}\beta^*_{kl,m}, \tag{9.5}$$

$$e^P_{kl,m} + \epsilon_{klq}\kappa^P_{mq} = \beta^*_{kl,m}. \tag{9.6}$$

With these relations we can alternatively use the methods of section 4 to find any other desired results

9.1. Geometry

We find the dislocation density from (3.4) with (9.2) or (4.1) with (9.6)

$$\alpha_{pl}(\mathbf{r}) = -\epsilon_{pmk}\beta^*_{kl,m}$$

$$= \epsilon_{pmk}\delta_{k,m}(S)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\} + \epsilon_{pmk}\delta_k(S)\epsilon_{lqr}\Omega_q\delta_{rm}$$

$$= \delta_p(L)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\} + \delta_l(S)\Omega_p - \delta_{pl}\delta_k(S)\Omega_k. \tag{9.7}$$

Here the second equality follows from (9.3), and the third from Stokes' theorem (B26) and the expansion (A3). We see now that the dislocation model is a dislocation distribution consisting of the following two parts: First a dislocation line along L, which is the same as for the corresponding discrete defect line, (5.19), and second a constant dislocation distribution over the surface S. In other words it is a dislocation wall at the surface S, which terminates on a dislocation line at its boundary L. The disclination density vanishes from (4.2) and (9.5)

$$\theta_{pq} = -\epsilon_{pmk}1/2\epsilon_{lqr}\beta^*_{rl,km} = 0, \tag{9.8}$$

as expected.

We see that the continuity equation (3.6) or (4.3) is satisfied

$$\alpha_{pl,p}(\mathbf{r}) = \delta_{p,p}(L)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\} + \delta_p(L)\epsilon_{lqp}\Omega_q + \delta_{l,p}(S)\Omega_p - \delta_{k,l}(S)\Omega_k$$

$$= 0, \tag{9.9}$$

where we have used (B27–28).

We next find the total Burgers vector from (4.5) with (9.4–5)

$$B_l = -\oint_\lambda (\beta^*_{(kl)} - \beta^*_{[rl],k}x_r)\,dL_k$$

$$= -\oint_\lambda \beta^*_{kl}\,dL_k$$

$$= \oint_\lambda \delta_k(S)\{b_l + \epsilon_{lqr}\Omega_q(x_r - x_r^0)\}\,dL_k$$

$$= b_l + \epsilon_{lqr}\Omega_q(x_r^{S\lambda} - x_r^0). \tag{9.10}$$

Here the second equality follows by a partial integration, the third from (9.3), and the fourth from (B15–16), where $x_r^{S\lambda}$ is the point where λ crosses S (fig. 1).

9.2. Statics

If we substitute (9.3–4) into (2.15) we find for the displacement

$$u_n^T(\mathbf{r}) = \int_S C_{ijkl}G_{jn,i}(R)\{b_l + \epsilon_{lqr}\Omega_q(x'_r - x_r^0)\}\,dS'_r, \tag{9.11}$$

which is identical to (5.33). So we see that the displacement of the dislocation model is exactly the same as the displacement of the corresponding discrete defect line. Hence (5.34) follows in exactly the same way

$$u^T_{n,m}(\mathbf{r}) = \oint_L \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})\{b_l + \epsilon_{lqr}\Omega_q(x'_r - x^\circ_r)\}dL'_p + \int_S \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})\Omega_p dS'_l + \beta^*_{mn}(\mathbf{r}). \tag{9.12}$$

So we find the elastic distortion from (3.9) and (9.2)

$$\beta_{mn}(\mathbf{r}) = \oint_L \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})\{b_l + \epsilon_{lqr}\Omega_q(x'_r - x_r)\}dL'_p + \int_S \epsilon_{pmk} C_{klmn} G_{jn,i}(\mathbf{R})\Omega_p dS'_l \tag{9.13}$$

This relation could also be obtained from (3.17) and (9.7). It follows from the discussion in section 3.1.2 that this elastic distortion is a state quantity for the dislocation model, but according to the discussion of section 5.3 it is not a state quantity for the discrete defect line. This may be the physical significance of Mura's "elastic distortion" for a discrete disclination.

$$\begin{aligned} e_{mn}(\mathbf{r}) = &\oint_L \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})\{b_l + \epsilon_{lqr}\Omega_q(x'_r - x^\circ_r)\}dL'_{p(mn)} \\ &+ \int_S \epsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R})\Omega_p dS'_{l(mn)}. \end{aligned} \tag{9.14}$$

This relation also follows from (4.14) with (9.4) and (9.12), or (4.29) with (9.7–8). We see that this expression is identical to (5.36). So the elastic strain and hence stress of the dislocation model is identical to that of the corresponding defect line, as we stated before.

We next find the bend-twist. From (9.12) we have

$$\begin{aligned} u^T_{n,ms}(\mathbf{r}) = &\oint_L \epsilon_{pmk} C_{ijkl} G_{jn,is}(\mathbf{R})\{b_l + \epsilon_{lqr}\Omega(x'_r - x^\circ_r)\}dL'_p \\ &+ \int_S \epsilon_{pmk} C_{ijkl} G_{jn,is}(\mathbf{R})\Omega_p dS'_l + \beta^*_{mn,s}(\mathbf{r}), \end{aligned} \tag{9.15}$$

and from (4.15) and (9.5)

$$\begin{aligned} \kappa_{st}(\mathbf{r}) = 1/2 &\oint_L \epsilon_{tmn}\epsilon_{pmk} C_{ijkl} G_{jn,is}(\mathbf{R})\{b_l + \epsilon_{lqr}\Omega_q(x'_r - x^\circ_r)\}dL'_p \\ &+ 1/2 \int_S \epsilon_{tmn}\epsilon_{pmk} G_{jn,is}(\mathbf{R})\Omega_p dS'_l. \end{aligned} \tag{9.16}$$

This relation could also have been obtained from (3.14) with (9.13), or from (4.34) with (9.7–8).

We see from this section that there is a great similarity between the dislocation model and its corresponding discrete defect line. Therefore it is important to distinguish carefully between them. For example, the defect densities for the discrete line are given by (5.19–20), while for the dislocation model they are given by (9.7–8). So we see that in the transition the disclination density in (5.20) has been traded for the constant surface dislocation density in (9.7). As we saw both defects give the same elastic strain; explicit expressions for it can be obtained by substituting the above densities into (4.29). For the discrete defect line we find (5.37) and for the dislocation model (9.14) for the elastic strain. We see that the second term in (5.37) represents the contribution from the disclination density; it equals the second term in (9.14) which represents the contribution from the constant surface dislocation density. We shall illustrate some special cases of the dislocation model in future publications [25, 26].

Li and Gilman have also calculated the force on a discrete disclination line by assuming that it equals the force on the corresponding dislocation model. Since the present paper does not deal with forces a formal proof of this interesting result will be published elsewhere [23].

10. The Compensated Disclination Line

We now want to investigate a problem that complements the one of section 9. Consider a vanishing dislocation loop density and a disclination loop density given by (5.13)

$$\beta^*_{kl}=0, \tag{10.1}$$

$$\phi^*_{kq}(\mathbf{r})=-\delta_k(S)\Omega_q, \tag{10.2}$$

where Ω_q is a constant. This represents just a constant distribution of disclination loops on the surface S. Since there is no dislocation loop distribution, we shall find that there is no elastic strain for this defect. We find the plastic quantities from (5.14–16)

$$e^P_{kl}=0, \tag{10.3}$$

$$\kappa^P_{kq}=\phi^*_{kq}. \tag{10.4}$$

We now use the methods of section 4 to find any further desired results.

10.1. Geometry

The dislocation and disclination densities are from (4.1–2)

$$\begin{aligned}\alpha_{pl}(\mathbf{r})&=\epsilon_{pmk}\epsilon_{klq}\delta_m(S)\Omega_q\\&=\delta_{pl}\delta_k(S)\Omega_k-\delta_l(S)\Omega_p,\end{aligned} \tag{10.5}$$

$$\begin{aligned}\theta_{pq}(\mathbf{r})&=\epsilon_{pmk}\delta_{k,m}(S)\Omega_q\\&=\delta_p(L)\Omega_q,\end{aligned} \tag{10.6}$$

where we have used Stokes' theorem (B26). Therefore the defect in this case is a dislocation wall at the surface S, which terminates on a disclination at its boundary L. It consists of the same discrete disclination line as treated in section 5, (5.20), and a dislocation distribution on S, which is just right to make the elastic strain vanish. Therefore we have called it the *compensated disclination line*. Note that the sum of (10.5) and (9.7) gives (5.20), and that (10.6) is the same as (5.19). This was of course to be expected, because sections 9 and 10 represent a decomposition of the problem of section 5. We find that the above results satisfy the continuity equations (4.3–4)

$$\alpha_{pl,p}(\mathbf{r})+\epsilon_{lpq}\theta_{pq}(\mathbf{r})=\delta_{k,l}(S)\Omega_k-\delta_{l,p}(S)\Omega_p+\epsilon_{lpq}\delta_p(L)\Omega_q=0, \tag{10.7}$$

$$\theta_{pq,p}(\mathbf{r})=\delta_{p,p}(L)\Omega_q=0, \tag{10.8}$$

using (B27–28).

We next find the total Burgers vector from (4.5) and (10.2–4)

$$\begin{aligned}B_l&=-\oint_\lambda \epsilon_{lqr}\delta_k(S)\Omega_q x_r dL_k\\&=-\epsilon_{lqr}\Omega_q x_r^{S\lambda},\end{aligned} \tag{10.9}$$

by (B16), where $x_r^{S\lambda}$ is the point of intersection between the curve λ and the surface S (fig. 1). We note that the sum of (10.9) and (9.10) is the same as (5.17). We find the Frank vector from (4.6), (10.4), and (10.2)

$$-\oint_{\lambda} \kappa_{kq}^{P} dL_k = \oint_{\lambda} \delta_k(S)\Omega_q dL_k = \Omega_q, \tag{10.10}$$

using (B15). This identifies the constant Ω_q as the Frank vector. The same results can be obtained from (4.7–8) and (10.5–6).

10.2. Statics

If we substitute (10.3) into (2.15) we find the displacement

$$u_n^T = 0. \tag{10.11}$$

Hence from (4.14) and (10.3)

$$e_{kl} = 0. \tag{10.12}$$

So we see that there is no elastic strain and hence stress, as we discussed before. The elastic bend-twist is from (4.15), (10.11), (10.4), and (10.2)

$$\kappa_{kq}(\mathbf{r}) = -\kappa_{kq}^{P}(\mathbf{r}) = \delta_k(S)\Omega_q. \tag{10.13}$$

For consistency we show that this relation can also be derived another way. If we substitute (10.5–6) into (4.34) we find

$$\kappa_{st}(\mathbf{r}) = -1/2 \int_S \epsilon_{tmn}\epsilon_{pmk}C_{ijkl}G_{jn,is}(\mathbf{R})\Omega_p dS'_l + 1/2 \oint_L \epsilon_{tmn}\epsilon_{pmk}\epsilon_{qsl}C_{ijkl}G_{jn,i}(\mathbf{R})\Omega_p dL'_q, \tag{10.14}$$

which by Stokes' theorem (A2), (2.7), and (B5) becomes

$$\begin{aligned}\kappa_{st}(\mathbf{r}) &= -1/2 \int_S \epsilon_{tmn}\epsilon_{pmk}C_{ijkl}G_{jn,il}(\mathbf{R})\Omega_p dS'_s \\ &= 1/2 \int_S \epsilon_{tmn}\epsilon_{pmk}\delta_{kn}\delta(\mathbf{R})\Omega_p dS'_s \\ &= \delta_s(S)\Omega_t,\end{aligned} \tag{10.15}$$

in agreement with (10.13). Note that the sum of (10.14) and (9.16) gives (5.40), as expected.

Summarizing, we see that when the compensated disclination line is added to the dislocation model we obtain the discrete defect line.

11. Summary

We started this paper with the general solution of the plastic strain problem which is just an extension of Eshelby's transformation problem and essentially equivalent to Mura's plastic distortion problem. It formed the basis of all static defect fields. We then reviewed dislocation theory, including the continuous distribution, the discrete line, Kroupa's continuous distribution of infinitesimal loops, and the dipole. This introductory material formed the point of departure for the general theory of defects, i.e., disclinations and dislocation combined. On the other hand, it formed a basis of comparison, because the general defect theory reduces to it when the disclinations vanish.

We defined the continuous distribution of defects. This definition was motivated by a violation of the compatibility equations. We also defined the Frank vector, which is the characteristic rotation vector of the disclinations, analogous to the Burgers vector for dislocations. We derived closed integral expressions for the basic elastic fields, the elastic strain and bend-twist, in terms of the defect densities, showing that they are state quantities. These integrals contain kernels with Green's tensor and the incompatibility source tensor, a type of Green's tensor introduced by Simmons and Bullough.

The definition of the discrete defect line was motivated by Weingarten's theorem. A new quantity introduced here is the axis of rotation, which did not exist for a continuous distribution. After finding the basic plastic fields, the plastic strain and bend-twist, all the results for the continuous distribution can be specialized to the discrete case. We found that the calculations were simplified by using two new quantities, Mura's "plastic distortion" and "plastic rotation," which we later identified as the dislocation and disclination loop densities. We found that the axis of a disclination can be translated to a new position by adding a discrete dislocation line to it. We found the basic elastic fields as closed line integral, which confirmed again their nature as state quantities.

The continuous distribution of infinitesimal defect loops was defined by extending Kroupa's definition for dislocation loops. The complete correlation to the continuous distribution of defects was established by finding the basic plastic fields in terms of the loop densities. It was then simple to identify Mura's plastic quantities, as mentioned above, which resolves a difference we had with Mura.

We found that the three independent definitions for defects, i.e., for continuous distributions, discrete lines, and infinitesimal loops, do not lead to a single concept of the disclination. Rather, the disclination defined in one formulation contains a certain amount of dislocation in the two other formulations. So a certain amount of mixing occurs in going from one formulation to another, but the amount can be uniquely determined. Our difference with Anthony for example originates from this mixing between continuous defect distributions and discrete defect lines. As another example we interpreted the discrete defect line in terms of a continuous distribution of defect loops, which clarifies our difference with Mura.

The discrete dipole line was defined by extending Kroupa's definition of the dislocation dipole. We find there is a great similarity between disclination dipoles and dislocation lines.

Then we showed that the general defect theory reduces to the classical dislocation theory when the disclinations vanish.

We clarified the concept of the "dislocation model" of a discrete defect line, introduced by Li and Gilman. We found that it is a distribution of dislocation loops identical to that for the corresponding discrete defect, but without any distribution of disclination loops. In Mura's terms it is given by the "plastic distortion" without "plastic rotation." We found that the dislocation model is a dislocation wall which terminates on a discrete dislocation lying along the same line as the corresponding defect line. The dislocation model has the same total displacement, elastic strain, and stress as the corresponding discrete defect line. The "elastic distortion" of the dislocation model is Mura's "elastic distortion" of the discrete defect line. The great similarity between the dislocation model and its corresponding discrete defect line makes it important to distinguish clearly between these two concepts.

We concluded the paper with the "compensated disclination line," which is a constant distribution of disclination loops over a surface. In Mura's terms it is given by the "plastic rotation" without "plastic distortion." We found that it is a dislocation wall terminating at a discrete disclination line, giving no displacement, strain, and stress. The sum of the compensated disclination line and the dislocation model gives the discrete defect line.

In an appendix we have developed a special notation, adapted from Kunin. It is very helpful for the treatment of discrete defects where generalized functions appear, such as the Dirac delta function.

So we have presented a general theory of stationary defects for a linearly elastic, infinitely extended, homogeneous body. In future publications we shall specialize these results to isotropy [24] and apply them to straight disclinations [25] and disclination loops [26].

The major shortcoming of the present treatment might be the use of linear theory. This means that in a real solid the resulting fields close to discrete defects will deviate considerably from our formulas, but they will become more realistic the further away we are from a defect. This point will be more clearly illustrated in the future publications where we obtain specific results for particular geometries. However, without the linear assumption we certainly could not have pushed the theory as far as we did. This is the price we paid for a fairly complete analytic treatment.

Within its limitations the present theory is completely self-consistent. Aside from its possible intrinsic usefulness, it can be used as a starting point for generalizations, such as dynamics, nonlinear effects, couple-stresses, a finite body, or inhomogeneities.

12. Appendix A. The Divergence Theorem and Stokes' Theorem

The *rank* (also called *order* by some authors) of a tensor equals the number of subscripts on the tensor. In the following let T be a tensor of any rank, where we have suppressed the subscripts.

The *divergence theorem* is formulated as follows

$$\int_V T_{,i} dV = \oint_S T dS_i, \tag{A1}$$

where the integrations are restricted to the arbitrary volume V and its bounding surface S, which is therefore a closed surface.

Stokes' theorem can be formulated in two ways. The first one is

$$\int_S \epsilon_{ijk} T_{,j} dS_i = \oint_L T dL_k, \tag{A2}$$

where ϵ_{ijk} is the permutation symbol. By using the identity

$$\epsilon_{ijk}\epsilon_{klm} \equiv \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{A3}$$

we find the second formulation

$$\oint_L \epsilon_{klm} T dL_k = \int_S (T_{,m} dS_l - T_{,l} dS_m). \tag{A4}$$

Here the integrations are restricted to the arbitrary surface S and its bounding curve L, which is therefore a closed curve. We use the right-hand rule in relating the curve L to the surface S.

13. Appendix B. Delta Functions on Curves and Surfaces

This appendix is an adaptation of a treatment by Kunin [27]. Let $\phi(x)$ be an infinitely differentiable finite function of x, called a test function. We can define the *Dirac delta function* $\delta(x)$ by

$$\int_a^b \delta(x-x')\phi(x)dx = \begin{cases} 0, & \text{if } x' < a, \\ \phi(x'), & \text{if } a < x' < b, \\ 0, & \text{if } b < x', \end{cases} \tag{B1}$$

where a and b are arbitrary constants. Since $\delta(x)$ is a generalized function, the integrals in this and subsequent equations are meaningless in the sense of classical analysis. Instead they are to be regarded as a symbolic notation. This notation has been treated in detail by Gel'fand and Shilov [28]. The meaning to be attached to the following integrals then follows from the fact that all subsequent results are derived from the definition (B1).

The three-dimensional Dirac delta function $\delta(\mathbf{r})$ is defined by

$$\delta(\mathbf{r}-\mathbf{r}') \equiv \delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3')\cdot \tag{B2}$$

Let the test function $\phi(\mathbf{r})$ be an infinitely differentiable finite function of position $\mathbf{r}$. Then it follows from (B1–2) that

$$\int_V \delta(\mathbf{r}-\mathbf{r}')\phi(\mathbf{r})dV = \begin{cases} \phi(\mathbf{r}'), & \text{if } \mathbf{r}' \text{ is in } V, \\ 0, & \text{if } \mathbf{r}' \text{ is not in } V, \end{cases} \tag{B3}$$

where $dV = dx_1dx_2dx_3$ and V is an arbitrary volume.

The Dirac delta functions for a curve L, a surface S, or a volume V are defined by

$$\delta_i(L) \equiv \int_L \delta(\mathbf{r}-\mathbf{r}')dL_i', \tag{B4}$$

$$\delta_i(S) \equiv \int_S \delta(\mathbf{r}-\mathbf{r}')dS_i', \tag{B5}$$

$$\delta(V) \equiv \int_V \delta(\mathbf{r}-\mathbf{r}')dV'\cdot \tag{B6}$$

We see that $\delta_i(L)$ and $\delta_i(S)$, in addition to being delta functions, are also vectors parallel to the curve L and normal to the surface S, respectively. From (B3) we see that

$$\delta(V) = \begin{cases} 1, & \text{if } \mathbf{r} \text{ is in } V, \\ 0, & \text{if } \mathbf{r} \text{ is not in } V. \end{cases} \tag{B7}$$

We now have the following relations

$$\int \delta_i(L)\phi(\mathbf{r})dV = \int_L \phi(\mathbf{r})dL_i, \tag{B8}$$

$$\int \delta_i(S)\phi(\mathbf{r})dV = \int_S \phi(\mathbf{r})dS_i, \tag{B9}$$

$$\int \delta(V)\phi(\mathbf{r})dV = \int_V \phi(\mathbf{r})dV, \tag{B10}$$

where the integrals in the left-hand sides are over all space, and those on the right-hand sides are restricted to the curve L, surface S, and volume V, respectively. To prove the first relation, (B8), substitute (B4) in the left-hand side, and we have

$$\int\int_L \delta(\mathbf{r}-\mathbf{r}')dL_i'\phi(\mathbf{r})dV = \int_L\int \delta(\mathbf{r}-\mathbf{r}')\phi(\mathbf{r})dVdL_i' = \int_L \phi(\mathbf{r}')dL_i'$$

by interchanging integration and using (B3). The proofs of (B9) and (B10) follow in a similar way. By a slight change of variable we can also write (B8–10) as follows

$$\int \delta_i(L')\phi(\mathbf{r}-\mathbf{r}')dV' = \int_l \phi(\mathbf{r}-\mathbf{r}')dL'_i, \tag{B11}$$

$$\int \delta_i(S')\phi(\mathbf{r}-\mathbf{r}')dV' = \int_S \phi(\mathbf{r}-\mathbf{r}')dS'_i, \tag{B12}$$

$$\int \delta(V')\phi(\mathbf{r}-\mathbf{r}')dV' = \int_V \phi(\mathbf{r}-\mathbf{r}')dV'. \tag{B13}$$

If the curve L crosses the surface S once, we also have the relation

$$\int_S \int_L \delta(\mathbf{r}-\mathbf{r}')dL'_i dS_j = \frac{t_i n_j}{|t_k n_k|}, \tag{B14}$$

where t_i is the unit tangent to the curve L, and n_j the unit normal to the surface S, at the point of intersection, $\mathbf{r}^{LS}$. To prove this relation, note that the integrals in (B14) contribute only at the point $\mathbf{r}^{LS}$. Therefore we can replace the curve L by a straight line tangent to L at $\mathbf{r}^{LS}$, and the surface S by a plane tangent to S at $\mathbf{r}^{LS}$. We shall next calculate the integral in (B14) for the special case $n_j=(001)$, i.e., when S is the x_1x_2 plane. For $j=1$, 2 the integrals then vanish in agreement with (B14). For $j=3$ we have by (B2) and (B1)

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_L \delta(x_1-x'_1)\delta(x_2-x'_2)\delta(x_3-x'_3)dL'_i dx_1 dx_2 &= \int_L \delta(x_3-x'_3)dL'_i \\ &= \int_{-\infty}^{\infty} \delta(x_3-x'_3)\frac{t_i}{|t_3|}dx'_3 \\ &= \frac{t_i}{|t_3|}.\end{aligned}$$

Here we have used the fact that for a straight line $dL'_i=(t_i/t_3)dx'_3$, and that the range of x'_3 is $(-\infty, \infty)$ for t_3 positive and $(\infty, -\infty)$ for t_3 negative. This result also agrees with (B14) for this special case. Hence (B14) holds in a particular coordinate system. By tensor analysis it is therefore also true in a general coordinate system. The specific form of (B14) can be derived from the above results by the method of Appendix II of reference [29]. From (B4), (B5), and (B14) we have

$$\int_S \delta_i(L)dS_i = \int_L \delta_i(S)dL_i = \begin{cases} 1, & \text{if } L \text{ crosses } S \text{ positively,} \\ 0, & \text{if } L \text{ does not cross } S, \\ -1, & \text{if } L \text{ crosses } S \text{ negatively.}\end{cases} \tag{B15}$$

It is not difficult to generalize these relations to

$$\int_S \delta_i(L)\phi(\mathbf{r})dS_i = \int_L \delta_i(S)\phi(\mathbf{r})dL_i = \begin{cases} \phi(\mathbf{r}^{LS}), & \text{if } L \text{ crosses } S \text{ positively,} \\ 0, & \text{if } L \text{ does not cross } S, \\ -\phi(\mathbf{r}^{LS}), & \text{if } L \text{ crosses } S \text{ negatively.}\end{cases} \tag{B16}$$

The *derivative* of delta functions is defined by switching the operation to the test function

$$\int_V \delta_{,j}(\mathbf{r}-\mathbf{r}')\phi(\mathbf{r})dV \equiv -\int_V \delta(\mathbf{r}-\mathbf{r}')\phi_{,j}(\mathbf{r})dV, \tag{B17}$$

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as suggested by classical analysis. Now we can write the derivatives of the delta functions (B4–6) as follows

$$\delta_{i,j}(L)=\int_L \delta_{,j}(\mathbf{r}-\mathbf{r}')dL_i', \tag{B18}$$

$$\delta_{i,j}(S)=\int_S \delta_{,j}(\mathbf{r}-\mathbf{r}')dS_i', \tag{B19}$$

$$\delta_{,j}(V)=\int_V \delta_{,j}(\mathbf{r}-\mathbf{r}')dV'. \tag{B20}$$

From these relations, it follows that

$$\int \delta_{i,j}(L)\phi(\mathbf{r})dV=-\int_L \phi_{,j}(\mathbf{r})\ dL_i, \tag{B21}$$

$$\int \delta_{i,j}(S)\phi(\mathbf{r})dV=-\int_S \phi_{,j}(\mathbf{r})dS_i, \tag{B22}$$

$$\int \delta_{,j}(V)\phi(\mathbf{r})dV=-\int_V \phi_{,j}(\mathbf{r})dV. \tag{B23}$$

For example, to prove (B21), substitute (B18) into the left hand side, and we have by (B17) and (B3)

$$\begin{aligned}\int\int_L \delta_{,j}(\mathbf{r}-\mathbf{r}')dL_i'\phi(\mathbf{r})dV &= \int_L\int \delta_{,j}(\mathbf{r}-\mathbf{r}')\phi(\mathbf{r})dVdL_i' \\ &= -\int_L\int \delta(\mathbf{r}-\mathbf{r}')\phi_{,j}(\mathbf{r})dVdL_i' \\ &= -\int_L \phi_{,j'}(\mathbf{r}')dL_i'.\end{aligned}$$

The proofs of (B22) and (B23) follow in a similar way.

The *divergence theorem* also holds for delta functions and is expressed as follows

$$\delta_{,i}(V)=-\delta_i(S), \tag{B24}$$

where S is the closed surface which is the boundary of V. This theorem is proved showing the following relationship:

$$\begin{aligned}\int \delta_{,i}(V)\phi(\mathbf{r})dV &= -\int_V \phi_{,i}(\mathbf{r})dV \\ &= -\oint_S \phi(\mathbf{r})dS_i \\ &= -\int \delta_i(S)\phi(\mathbf{r})dV,\end{aligned}$$

which follows from (B23), (A1), and (B9). Since $\phi(\mathbf{r})$ can be chosen arbitrarily, (B24) follows. We also conclude from this relation that

$$\epsilon_{ijk}\delta_{k,j}(S)=0 \tag{B25}$$

for a closed surface S.

Furthermore *Stokes' theorem* also holds for delta functions as follows

$$\epsilon_{ijk}\delta_{i,j}(S)=-\delta_k(L), \tag{B26}$$

or

$$\epsilon_{klm}\delta_k(L)=\delta_{m,l}(S)-\delta_{l,m}(S), \tag{B27}$$

where L is the closed curve which is the boundary of S. This theorem is proved by showing the following relationship:

$$\begin{aligned}\int \epsilon_{ijk}\delta_{i,j}(S)\phi(\mathbf{r})dV &= -\int_S \epsilon_{ijk}\phi_{,j}(\mathbf{r})dS_i \\ &= -\oint_L \phi(\mathbf{r})dL_k \\ &= -\int \delta_k(L)\phi(\mathbf{r})dV,\end{aligned}$$

which follows from (B22), (A2), and (B8). Since $\phi(\mathbf{r})$ can be chosen arbitrarily, (B26) follows, and (B27) follows directly from (B26). We conclude from (B26) that

$$\delta_{k,k}(L)=0 \tag{B28}$$

for a closed curve L.

A *homogeneous function* of degree λ is defined by the equation

$$f(kx)=k^{\lambda}f(x) \tag{B29}$$

for any positive k. We wish to show that $\delta(\mathbf{r})$ is a homogeneous function of degree (-3) in r. From (B3) we have

$$\int \delta(\mathbf{r})\phi(\mathbf{r})dV=\phi(0). \tag{B30}$$

By a change of variable and (B3) we have

$$\begin{aligned}\int \delta(k\mathbf{r})\phi(\mathbf{r})dV &= k^{-3}\int \delta(\mathbf{r})\phi(k^{-1}\mathbf{r})dV \\ &= k^{-3}\phi(0).\end{aligned} \tag{B31}$$

Therefore, since $\phi(\mathbf{r})$ is arbitrary

$$\delta(k\mathbf{r})=k^{-3}\delta(\mathbf{r}), \tag{B32}$$

which shows the contention.

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