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Product Solutions and Separation of Variables

A. J. Goldman

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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This paper consists of two independent notes, whose common features are (a) their concern with "product-form" functions and (b) their use of an abstract-algebra setting to isolate the essential features of the problems treated. The first note further generalizes the generalizations given by Chu and Diaz (1965) of their observation that Euler's difference equation $y(x+1)-y(x)=f(x)$, when f is of period 1, has the product-form solution $y(x)=xf(x)$. The second note formalizes and rigorously proves the fact that a function is separable jointly in its variables iff it is separable in each individual variable.

Key words: Difference equations, functional equations, separation of variables.

1. Product Solutions of Certain Functional Equations

For this paragraph only, let $f(x)$ be a real function of period 1. Chu and Diaz,[1] apparently for the first time, observed that Euler's difference equation

$$y(x+1)-y(x)=f(x)$$

admits the simple solution $y(x)=xf(x)$. As a first generalization, they noted that the linear difference equation

$$a_n(x)y(x+n)+\ldots+a_0(x)y(x)=f(x)$$

admits the solution $y(x)=g(x)f(x)$, where $g(x)$ is any solution of

$$a_n(x)g(x+n)+\ldots+a_0(x)g(x)=1.$$

As a next generalization, they observed that the difference equation

$$F(y(x), y(x+1), \ldots, y(x+n))=f(x),$$

where F is a homogeneous function of degree 1 of its $n+1$ arguments, has the solution $y(x)=g(x)f(x)$ where $g(x)$ is any solution of

$$F(g(x), g(x+1), \ldots, g(x+n))=1.$$

In this note we seek to isolate the principle involved. Let S and S' be multiplicative semigroups, $\theta: S\rightarrow S'$ a function, and $F: S^{n+1}\rightarrow S'$ a function obeying the generalized homogeneity condition

$$F(g_0h, g_1h, \ldots, g_nh)=F(g_0, g_1, \ldots, g_n)\theta(h)$$

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[1] S. C. Chu and J. B. Diaz, On the Difference Equation $y(x+1)-y(x)=f(x)$, J. Math. Anal. Appl. 11 (1965), 478–481.

for all $g_0, g_1, \ldots, g_n, h \epsilon S$.

THEOREM: *Let* $T_0, T_1, \ldots, T_n$ *be homomorphisms of* S *which have a common fixed point* $f \epsilon S$. *Then the equation*

$$F(T_0 y, T_1 y, \ldots, T_n y) = c\theta(f) \tag{1}$$

admits the solution $y = gf$, *where* g *is any solution of*

$$F(T_0 g, T_1 g, \ldots, T_n g) = c \epsilon S'. \tag{2}$$

The proof is by simple calculation:

$$\begin{aligned}
&F(T_0 gf, T_1 gf, \ldots, T_n gf)\\
&= F((T_0 g)(T_0 f), (T_1 g)(T_1 f), \ldots, (T_n g)(T_n f))\\
&= F((T_0 g) f, (T_1 g) f, \ldots, (T_n g) f)\\
&= F(T_0 g, T_1 g, \ldots, T_n g)\theta(f)\\
&= c\theta(f).
\end{aligned}$$

To obtain the situation discussed at the end of the first paragraph, one need only take

$S = S'$ = semigroup of real functions of a real variable ("pointwise" multiplication),

θ = identity map of S,

$c = 1 \epsilon S$,

T = "step-up" operator, defined by $(Tg)(x) = g(x+1)$,

$T_i = T^i$ for $0 \leqslant i \leqslant n$.

2. On Separation of Variables

The method of separation-of-variables figures prominently in the solution of many problems of applied mathematics and mathematical physics. In this note we formalize and prove a simple principle, concerning this method, which appears to be frequently invoked on an implicit rather than explicit basis.

Let $X_1, \ldots, X_n$ be nonempty sets. We write $x = (x_1, \ldots, x_n)$ for a general member of the Cartesian product $X_1 \times \ldots \times X_n = X$, let X^i be the corresponding product with the factor X_i removed, and set $x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Let S be a commutative semigroup written multiplicatively, and $f: X \to S$ a function. We will call f *totally separable* (i.e., separable jointly in its variables) if there exist functions $f_i: X_i \to S$ for which

$$f(x) = f_1(x_1) \ldots f_n(x_n) \qquad (\text{all } x \epsilon X). \tag{3}$$

If there exist functions $g_i: X_i \to S$ and $h^i: X^i \to S$ such that

$$f(x) = g_i(x_i) h^i(x^i) \qquad (\text{all } x \epsilon X), \tag{4}$$

we call f *separable in the ith variable.* Clearly, if f is totally separable, then it is separable in each of its variables. The "simple principle" alluded to above, is that the converse of the last statement also holds.

More specifically, we call $(g_1, \ldots, g_n)$ an *associate* of f if there exists $(h^1, \ldots, h^n)$ such that (4) holds for all i, and we will prove that under these circumstances

$$f(x) = cg_1(x_1) \ldots g_n(x_n) \qquad (\text{all } x \epsilon X) \tag{5}$$

for some $c \epsilon S$. The proof is simplest when $S = G$, a multiplicative *group*. This restriction would however rule out direct applicability to the most important case in which S is the real axis R. We could take $G = R - \{0\}$ and then perhaps handle zero values by appeal to some further "smoothness" assumption, but this is inelegant; instead we will prove the result for the case $S = G \cup \{0\}$, where 0 is the unique member of $S - G$ and is such that $s.0 = 0$ for all $s \epsilon S$. (Stripping out appropriate parts of the proof will establish the result when $S = G$.)

The proof will be by induction on n. The result is clearly true for $n = 1$, since then the function h^1 of (4) is a constant c, so that $f(x_1) = cg_1(x_1)$.

Suppose the result is true for $n = k$, that $X = X_1 \times \ldots \times X_k \times X_{k+1}$, and that $f: X \rightarrow S$ has associate $(g_1, \ldots, g_{k+1})$ with corresponding $(h^1, \ldots, h^{k+1})$ as in (4). Set $Y_i = X_i - g_i^{-1}(0)$. If any Y_i is empty, then (4) shows that $f = 0$, and so (5) holds with $c = 0$. It may therefore be assumed that each Y_i is nonempty; let $Y = Y_1 \times \ldots \times Y_{k+1}$. We will prove the existence of $c \epsilon S$ for which (5) holds for all $x \epsilon Y$; it will then, by (4), automatically hold for all $x \epsilon X - Y$ as well.

For $1 \leqslant i \leqslant k$,

$$f(x) = g_{k+1}(x_{k+1}) h^{k+1}(x^{k+1}) = g_i(x_i) h^i(x^i),$$

and so

$$[g_{k+1}(x_{k+1})]^{-1} h^i(x^i) = [g_i(x_i)]^{-1} h^{k+1}(x^{k+1}) \qquad (\text{all } x \epsilon Y).$$

The left-hand side is independent of x_i and the right-hand side is independent of x_{k+1}, so that each is equal to some $G_i(x)$ which is independent of both x_i and x_{k+1}. If we set $Y' = Y_1 \times \ldots \times Y_k$ and write $x' = (x_1, \ldots, x_k)$, then

$$f(x) = f(x'; x_{k+1}) = g_{k+1}(x_{k+1})\ g_i(x_i)\ H_i(x') \qquad (\text{all } x \epsilon Y),$$

for $1 \leqslant i \leqslant k$, where $H_i(x') = G_i(x)$ is independent of x_i. Hence for each $x_{k+1} \epsilon Y_{k+1}$, the function $F: Y' \rightarrow S$ defined by

$$F(x') = [g_{k+1}(x_{k+1})]^{-1} f(x'; x_{k+1})$$

has $(g^1, \ldots, g^k)$ as associate;[2] by the induction hypothesis, there is a $c(x_{k+1}) \epsilon S$ such that

$$F(x') = g_1(x_1) \ldots g_k(x_k)\ c(x_{k+1}) \qquad (\text{all } x' \epsilon Y')$$

and so

$$f(x) = g_1(x_1) \ldots g_{k+1}(x_{k+1}) c(x_{k+1}) \qquad (\text{all } x \epsilon Y).$$

But by (4),

$$f(x) = g_{k+1}(x_{k+1})\ h^{k+1}(x^{k+1}) \qquad (\text{all } x \epsilon Y),$$

[2] More precisely, the vector of function-restrictions $g_i|Y_i$ for $1 \leqslant i \leqslant k$.

so that

$$c(x_{k+1}) = [g_1(x_1) \quad . \quad . \quad . \quad g_k\ (x_k)]^{-1}\ h^{k+1}\ (x^{k+1}).$$

Since the left-hand side depends only on x_{k+1} while the right-hand side is independent of this variable, $c(x_{k+1})$ must be a constant, and so (5) is established for all $x \epsilon Y$, completing the proof.

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