JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematical Sciences
Vol. 76B, Nos. 3 and 4, July–December 1972

On Regular Sets of Polynomials Whose Zeros Lie in Prescribed Domains

M. A. Farag* and M. Nassif*

(July 18, 1972)

The relation between the mode of increase of regular basic sets of finite span and the order of magnitude of the zeros of polynomials $\{p_n(z)\}$ belonging to them is investigated. Upper bounds are obtained for the order of the basic sets when the zeros of $p_n(z)$ lie either in the unit circle or in a circle whose radius increases in a certain manner with the index n of the polynomial.

Key words: Basic sets; Cannon sum of basic sets; lower semiblock matrices; order of basic sets; zeros of polynomials in regular sets.

1. Introduction

The relation between the mode of increase of simple sets[1] and the order of magnitude of the zeros of polynomials belonging to them has been the interest of many authors, of whom we may mention Eweida [1][2] and Nassif [3]. The same problem is studied in the present paper when the sets considered are *regular of finite span.* To formulate a precise definition of such sets we suppose that l is an integer greater than 1,[3] and the sequence (μ_n) of integers is constructed so that

$$\mu_0=0,\ 1\leqslant\mu_n-\mu_{n-1}\leqslant l \qquad ;\ (n\geqslant 1). \tag{1.1}$$

Thus, if we put

$$\nu_n=\mu_n-\mu_{n-1} \qquad ;\ (n\geqslant 1), \tag{1.2}$$

then

$$1\leqslant\nu_n\leqslant l \qquad ;\ n\leqslant\mu_n\leqslant nl. \tag{1.3}$$

Let $\{p_n(z)\}$ be a set of polynomials and let d_n be the degree of the polynomial $p_n(z)$, so that we can write

$$p_n(z)=\sum_{k=0}^{d_n}p_{n,k}z^k. \tag{1.4}$$

We shall assume that

$$d_0=0;\ m<d_m\leqslant\mu_n \quad \text{when} \quad \mu_{n-1}<m<\mu_n;\ d\mu_n=\mu_n;\ (n\geqslant 1). \tag{1.5}$$

AMS Subject Classification Numbers: Primary 3010.

*An invited paper. Present address, Mathematics Department, Assiut University, Assiut, Egypt.

[1] The reader is supposed to be acquainted with the theory of basic sets of polynomials as given by Whittaker [4, 5]. The mode of increase of a basic set is determined by its order and type, c. f. Whittaker [4; pp. 11, 12]. Our concern here is with the order.

[2] Figures in brackets indicate the literature references at the end of this paper.

[3] When $l=1$, the set will be simple.

When the set $\{p_n(z)\}$ thus defined, is basic it is said to be *regular of finite span of bound* l, and the set is normalized in the sense that

$$p_{n, d_n}=1 \qquad ; (n \geqslant 0). \tag{1.6}$$

The matrix $(p_{n,k})$ of coefficients of such sets is a lower semi-block[4] matrix whose leading diagonal consists of square matrices (Δ_n) of the form

$$\begin{cases}\Delta_0=(1)\\ \Delta_n=(p_{\mu_n-r, \mu_n-s});\ 0 \leqslant r,\ s \leqslant \nu_n-1;\ (n \geqslant 1).\end{cases} \tag{1.7}$$

Let δ_n denote the modulus of the determinant $|\Delta_n|$; δ_n should be positive to ensure that the set is basic. Write

$$\sigma=\liminf_{n \to \infty}\frac{\log \delta_1\delta_2 \ldots \delta_n}{n \log n}. \tag{1.8}$$

With the above notation, the main results of the present work are formulated in the following theorems.[5]

THEOREM 1: *Let* $\{\mathrm{p_n(z)}\}$ *be a regular set of polynomials of finite span of bound* l, *and suppose that the zeros of the polynomials* $\{\mathrm{p_n(z)}\}$ *all lie in* $|\mathrm{z}| \leqslant 1$. *Then the set will be of order not exceeding* $\frac{1}{2}(l+1)-\frac{\sigma}{l}$, *and this bound is attainable.*

THEOREM 2: *Let the set* $\{\mathrm{p_n(z)}\}$ *be as in Theorem* 1 *and suppose that the zeros of the polynomial* $\mathrm{p_n(z)}$ *lie in* $|\mathrm{z}| \leqslant \mathrm{n}^\alpha$, *where* α *is a positive finite number. Then the order of the set* $\{\mathrm{p_n(z)}\}$ *will not exceed* $\frac{1}{2}(\alpha+1)(l+1)-\frac{\sigma}{l}$.

2. Preliminary Results

We shall establish in this section a lemma, of general type, which is the basis for the proofs of the above theorems. In fact, we shall suppose that the zeros of the polynomial $p_n(z)$ lie in $|z| \leqslant \rho_n$, where the numbers (ρ_n) accord to the following restrictions.

$$\rho_{n+1} \geqslant \rho_n \geqslant 1 \qquad ; (n \geqslant 1), \tag{2.1}$$

and there is a finite number $a \geqslant 1$ for which

$$(\rho_{\mu_n}/\rho_{\mu_{n-1}})^{\mu_{n-1}} \leqslant a^{\nu_n} \qquad ; (n \geqslant 2). \tag{2.2}$$

In view of (1.4), (1.5), (1.6), and (2.1) it can be verified that

$$|p_{m,k}| \leqslant (k^{\mu_n})\rho_{\mu_n}^{\mu_n-k};\ (0 \leqslant k \leqslant d_m,\ \mu_{n-1} < m \leqslant \mu_n;\ n \geqslant 1). \tag{2.3}$$

Inserting (2.3) in (1.7) we find that

$$\delta_0=1;\ \delta_n < \lambda_n\rho_{\mu_n}^{1/2\nu_n(\nu_n-1)}=\lambda_n T_n\rho_{\mu_n}^{-\nu_n} \qquad ; (n \geqslant 1), \tag{2.4}$$

---

[4] c. f. Ibrahim [2; p. 282].

[5] It should be observed that a substitution $z=k_{z'}+b$ transforms the circles $|z-b|=k$, $|z-b|=kn^\alpha$ onto the respective circles $|z'|=1$, $|z'|=n^\alpha$! Also, according to Whittaker's theorem [4; p. 12] such substitution does not affect the order of the basic set. Hence there is no loss of generality in assuming the zeros to lie in $|z| \leqslant 1$ in Theorem 1 and in $|z| \leqslant n^\alpha$ in Theorem 2. We note also that, as far as the order is concerned, the results of Theorems 1 and 2 above reduce to those of Nassif [3; Theorems 6.1, 6.2] when $l=1$.

where

$$\lambda_0 = T_0 = 1,\ \lambda_n = \nu_n! \binom{\mu_n}{1}\binom{\mu_n}{2} \cdots \cdots \binom{\mu_n}{\nu_n - 1};\ T_n = \rho_{\mu_n}^{\nu_n(\nu_n+1)/2}\ ;\ (n \geqslant 1) \tag{2.5}$$

Suppose that z^n admits the representation

$$z^n = \sum_k \pi_{n,k} p_k(z). \tag{2.6}$$

The required lemma is concerned with the coefficients $(\pi_{n,k})$:

LEMMA 1: *Let the set* $\{p_n(z)\}$ *be as in Theorem* 1 *and suppose that the zeros of the polynomial* $p_n(z)$ *lie in* $|z| \leqslant \rho_n$. *Then the coefficients* $(\pi_{n,k})$ *satisfy the inequality*

$$|\pi_{\mu_n - i,\, \mu_k - j}| < \frac{i!(\mu_n - i)!\lambda_k\lambda_{k+1} \cdots \cdots \lambda_n,\ T_k T_{k+1} \cdots \cdots T_n}{\rho^i_{\mu_n}(\mu_k)!\delta_k\delta_{k+1} \cdots \delta_n} c^{\mu_n - \mu_k}, \tag{2.7}$$

for $0 \leqslant i \leqslant \nu_n - 1;\ 0 \leqslant j \leqslant \nu_k - 1;\ n \geqslant k \geqslant 0$, *where the constant* c *is fixed by*

$$c = \frac{a}{\log\ (1 + i/l)} > a \geqslant 1. \tag{2.8}$$

PROOF: It should be observed, first of all, that the matrix $(\pi_{n,k})$, which, according to (2.6), is the unique inverse of the matrix of coefficients $(p_{n,k})$, is of the same structure as this last matrix. Thus, carrying out the product of the matrices (Δ_k) with their corresponding matrices in $(\pi_{n,k})$, we get $\pi_{0,0} = 1$, and

$$(p_{\mu_k - r,\, \mu_k - s})(\pi_{\mu_k - t,\, \mu_k - u}) = I_k,\ (0 \leqslant r, s, t, u \leqslant \nu_k - 1;\ k \geqslant 1),$$

where I_k is the unit matrix of order ν_k. Applying (2.1), (2.3), (2.4), and (2.5) the following inequality is obtained.

$$|\Pi_{\mu_k - i,\, \mu_k - j}| < \frac{i!(\mu_{k-i})!\lambda_k T_k}{(\mu_k)!\rho^i_{\mu_k}\delta_k};\ (0 \leqslant i,\ j \leqslant \nu_k - 1;\ k \geqslant 1). \tag{2.9}$$

The product is then carried out with respect the remaining elements of the matrices $(p_{n,k})$ and $(\Pi_{n,k})$. When $k \geqslant 1$, the following equations are formed.

$$\sum_{s=\mu_{k-1}+1}^{\mu_n} p_{\mu_n - r,\, s}\Pi_{s,\, \mu_k - j} = 0;\ r = 0, 1, \ldots, \nu_n - 1;\ (0 \leqslant j \leqslant \nu_k - 1;\ n \geqslant k \geqslant 1). \tag{2.10}$$

These equations can be solved for the coefficients $(\Pi_{\mu_n - i,\, \mu_k - j})$; $i = 0, 1, \ldots, \nu_n - 1$, since $\delta_n > 0$. Appealing to the relations (2.1), (2.3), and (2.5) we obtain

$$|\Pi_{\mu_n - i,\, \mu_k - j}| \leqslant \frac{\lambda_n T_n}{\binom{\mu_n}{i}\rho^{i+\nu_n}_{\mu_n}\delta_n} \sum_{s=\mu_{k-1}+1}^{\mu_{n-1}} \binom{\mu_n}{s} \rho_{\mu_n}^{\mu_n - s} |\Pi_{s,\, \mu_k - j}|, \tag{2.11}$$

for $0 \leqslant i \leqslant \nu_n - 1;\ 0 \leqslant j \leqslant \nu_k - 1;\ n \geqslant k \geqslant 1$. The inequality (2.7) of the lemma will be deduced from (2.11) when $k \geqslant 1$. In fact, it is seen from (2.9) that (2.7) is true for $n = k$. Also, putting $n = k + 1$ in (2.11) and applying (1.3), (2.1), (2.2), (2.8), and (2.9) it can be verified that (2.7) is also satisfied for

$n=k+1$. Moreover, suppose that (2.7) is valid for $n=k, k+1, \ldots, m-1$, then by application of (1.2), (1.3), (2.1), (2.2), (2.4), (2.7), (2.8), and (2.11) and by simple calculation we can arrive at the following relation.

$$|\Pi_{\mu_m-i,\mu_k-j}| < \frac{T_m\lambda_m}{\binom{\mu_m}{i}\rho_{\mu_m}^{i+\nu_m}\delta_m} \sum_{n=k}^{m-1}\sum_{r=0}^{\nu_n-1}\binom{\mu_m}{\mu_n-r}\rho_{\mu_m}^{\mu_m-\mu_n+r}|\Pi_{\mu_n-r,\mu_k-j}|$$

$$< \frac{i!(\mu_m-i)!\lambda_k\lambda_{k+1}\ldots\lambda_m.\ T_kT_{k+1}\ldots T_m}{(\mu_k)!\rho_{\mu_m}^i\ \delta_k\delta_{k+1}\ldots\delta_m} c^{\mu_m-\mu_k}.$$

Hence the inequality (2.7) of the lemma is true for $n \geqslant k \geqslant 1$.

Now, when $k=0$, (and hence $j=0$), the eqs (2.10) assume the form

$$\sum_{s=0}^{\mu_n} P_{\mu_n-r,s}\Pi_{s,0}=0;\ (r=0, 1, \ldots, \nu_n-1;\ n>0).$$

Solving these equations for the coefficients $(\Pi_{\mu_n-i,0})$; $i=0, 1, \ldots \nu_n-1$, and proceeding in the same way as before, we easily obtain the inequality

$$|\pi_{\mu_n-i,0}| < \frac{i!(\mu_{n-i})!\lambda_1\lambda_2\ldots\lambda_n\cdot T_1T_2\ldots T_n}{\rho_{\mu_n}^i\delta_1\delta_2\ldots\delta_n} c_{\mu_n}, \tag{2.12}$$

for $0 \leqslant i \leqslant \nu_n-1$; $n>0$. Noting that $\lambda_0 = T_0 = \lambda_0 = 1$; $\mu_0 = 0$ it will be seen that (2.12) is merely the inequality (2.7) for $k=0$. The lemma is therefore established.

3. Proof of Theorem 1

We shall suppose here that the zeros of the polynomials $\{p_n(z)\}$ all lie in $|z| \leqslant 1$. If $M_n(r)$ denotes the maximum value of $|p_n(z)|$ in $|z| \leqslant r$; $r>0$, then from (1.5) and (1.6) we have

$$M_{\mu_k-j}(r) \leqslant (r+1)^{\mu_k} \quad ; (0 \leqslant j \leqslant \nu_k - 1;\ k \geqslant 1). \tag{3.1}$$

Moreover, putting[6] $\rho_n = 1$ in (2.4), (2.5), and (2.7) we find that

$$\begin{cases} \delta_n < \lambda_n < l\mu_n^{\nu_n(l-1)/2}, \\ |\pi_{\mu_n-i,\mu_k-j}| < \dfrac{i!(\mu_n-i)!\lambda_k\lambda_{k+1}\ldots\lambda_n}{(\mu_k)!\delta_k\delta_{k+1}\ldots\delta_n} c^{\mu_n-\mu_k}, \\ (0 \leqslant i \leqslant \nu_n-1;\ 0 \leqslant j \leqslant \nu_k-1;\ n \geqslant k \geqslant 0);\ c = \dfrac{1}{\log(1+1/l)}. \end{cases} \tag{3.2}$$

Therefore

$$\lambda_1\lambda_2\ldots\lambda_n < l^n\mu_n^{\nu_n(l-1)/2} \quad ; (n \geqslant 1). \tag{3.3}$$

In the usual notation, the Cannon sum for the set $\{p_n(z)\}$ is

[6] The restrictions (2.1) and (2.2) for the numbers (ρ_n) are satisfied in this case with $a=1$.

$$\omega_n(r) = \sum_k |\pi_{n,k}| M_k(r). \tag{3.4}$$

Introduction of (3.1), (3.2), and (3.3) in (3.4) easily leads to the following relation

$$\omega_{\mu_n - i}(r) < l! \frac{(\mu_n - i)!\mu_n^{\nu_n(l-1)/2}}{} l^n \cdot c^{\mu_n} \exp\left(\frac{r+1}{c}\right);$$

for $0 \leqslant i \leqslant \nu_n - 1$; $n \geqslant 1$. The order ω of the basic set $\{p_n(z)\}$ can be evaluated from this relation by application of (1.3) and (1.8), whereby we obtain

$$\omega \leqslant \frac{1}{2}(l+1) - \frac{\sigma}{l}, \tag{3.5}$$

as required by Theorem 1.

4. Example

To complete the proof of the theorem a basic set is constructed, according to the conditions of the theorem, such that its order $\omega = \frac{1}{2}(l+1) - \frac{\sigma}{l}$. The following lemma is first proved.

LEMMA 2: *When* $l \geqslant 2$, *the function* $\mathrm{E}(z) = \sum_{n=0}^{\infty} z^{nl}/(nl)!$ *has at least one zero inside the circle* $|z| = \{(1+1)!\}^{1/l}$.

PROOF: When $l=2$, $E(z) = \cosh z$, which obviously has the required property. Therefore we shall assume that $l \geqslant 3$ and put

$$z^l = t;\ E(z) = e(t);\ f(t) = 1 + \frac{t}{l!} + \frac{t^2}{(2l)!};\ e(t) = f(t) + r(t). \tag{4.1}$$

Now, it is easily seen that when $l \geqslant 3$, the function $f(t)$ has a zero in $-(l+1)! < t < 0$, and therefore $f(t)$ has at least one zero in the circle $|t| = (l+1)!$. Moreover, by actual calculation it can be verified from (4.1) that

$$|f(t)| > 3/2 \text{ on } |t| = (l+1)! \qquad ;\ (l \geqslant 3),$$

and that

$$\max_{|t| = (l+1)!} |r(t)| < 2/9 \qquad ;\ (l \leqslant 3).$$

Hence, the required result follows by an application of Rouche's theorem.

Now, it is easily seen that the following set $\{p_n(z)\}$ satisfies the conditions of Theorem 1.

$$p_0(z) = 1;\ p_{nl-r}(z) = (1 + \bar{\epsilon}^r z)^{nl};\ (0 \leqslant r \leqslant l-1;\ n \geqslant 1), \tag{4.2}$$

where $\epsilon = \exp(2i\pi/l)$; $l \geqslant 2$. It is also clear that the zeros of the polynomials all lie on $|z| = 1$. From (4.2) we have

$$p_{nl-r,k} = \binom{nl}{k} \bar{\epsilon}^{rk};\ (0 \leqslant k \leqslant nl;\ n \geqslant 1;\ 0 \leqslant r \leqslant l-1). \tag{4.3}$$

Whence, in the notation (2.4) and (2.5), we obtain

$$(4.4)\qquad \begin{cases} \delta_n = (l!)^{-1}\lambda_n 2^{l(l-1)/2} \prod_{J=1}^{l-1} |\sin (j\pi/l)|^j; \\ \lambda_n = l!\binom{nl}{1}\binom{nl}{2} \cdots \binom{nl}{l-1}. \end{cases}$$

Simple calculation based on (4.4) leads to the fact that $\sigma = \frac{1}{2} l(l-1)$. Hence, in view of (3.5), we have to prove that $\omega = 1$.

In fact, from the matrix product $(p_{n,k})(\pi_{n,k}) = I$ it follows that

$$(4.5)\qquad \sum_{k=0}^{nl} p_{nl-r,k}\pi_{k,0} = 0,\ (r=0, 1, \ldots. l-1;\ n \geq 1).$$

Inserting (4.3) in (4.5), adding the results corresponding to $r=0, 1, \ldots, l-1$, and putting

$$(4.6)\qquad b_k = \frac{\pi_{kl,0}}{(kl)!},\ (k \geq 1);\ b_0 = \pi_{0,0} = 1,$$

we are led to the following relations

$$(4.7)\qquad \sum_{k=0}^{n} \frac{b_{n-k}}{(kl)!} = 0,\ (n \geq 1).$$

Writing $E(z) = \sum_{n=0}^{\infty} z^{nl}/(nl)!$ and $G(z) = \sum_{n=0}^{\infty} b_n z^{nl}$, we see that (4.7) implies that $G(z) = 1/E(z)$. Hence, by Lemma 2, we infer that $G(z)$ is regular in $|z| = \rho$, where $\rho < \{(l+1)!\}^{1/l}$. That is to say

$$(4.8)\qquad \limsup_{n\to\infty} |b_n|^{1/n} = 1/\rho.$$

Finally, from (4.6) and (4.8) we can deduce that $\omega = 1$, and the proof of Theorem 1 is therefore complete.

5. Proof of Theorem 2

We now suppose that the zeros of the polynomial $p_n(z)$, belonging to the regular set $\{p_n(z)\}$, lie in $|z| \leq n^\alpha$, where α is a positive number. Hence, in the notation of (3.1), we see that

$$(5.1)\qquad M_0(r) = 1,\ M_{\mu_k - j}(r) \leq \{r + (\mu_k)^\alpha\}^{\mu_k};\ (0 \leq j \leq \nu_k - 1;\ k \geq 1).$$

Moreover, putting[7] $\rho_n = n^\alpha$ in (2.4) and (2.7) we obtain

$$(5.2)\qquad \begin{cases} \delta_n < \lambda_n \mu_n^{\alpha\nu_n(\nu_n-1)/2} \\ |\pi_{\mu_n-i,\mu_k-j}| < \dfrac{i!(\mu_{n-1})!\lambda_k\lambda_{k+1} \ldots. \lambda_n,\ U_k U_{k+1} \ldots. U_n}{(\mu_k)!\delta_k\delta_{k+1} \ldots \delta_n} c^{\mu_n - \mu_k} \\ (0 \leq i \leq \nu_n - 1;\ 0 \leq j \leq \nu_k - 1;\ n \geq k \geq 0), \end{cases}$$

where

[7] The restrictions (2.1) and (2.2) for the numbers (ρ_n) are still here satisfied with $a = e^\alpha > 1$.

$$c=\frac{e^{\alpha}}{\log(1+i/l)},\ U_k=\mu_k^{\alpha\nu_k(\nu_k+1)/2} \qquad ;\ (k\geqslant 1),$$

so that

$$U_1U_2\ .\ .\ .\ .\ .\ U_n<\mu_n^{\alpha\mu_n(l+1)/2}. \tag{5.3}$$

A combination of (3.3), (3.4), (5.1), (5.2), and (5.3) easily yields

$$\omega_{\mu_{n-i}}(r)<l!\exp\left\{\frac{e^{\alpha}\,(r+1)}{c}\right\}l_{\mu_n c}^{n\alpha l}\mu_n\frac{(\mu_{n-i})!\mu_n^{\mu_n\{\alpha(l+1)+l-1\}/2}}{\delta_1\delta_2\ .\ .\ .\ \delta_n}, \tag{5.4}$$

for $0\leqslant i\leqslant \nu_n-1$; $n\geqslant 1$. The inequality of Theorem 2, namely

$$\omega\leqslant\frac{1}{2}(\alpha+1)\,(l+1)-\frac{\sigma}{l},$$

for the order of the basic set $\{p_n(z)\}$, can be easily derived from the relation (5.4). Theorem 2 is therefore established.

6. References

[1] Eweida, M. T., Duke Math J. **14**, 865–875 (1947).

[2] Ibrahim, S. A., J. of the London Math. Soc. **34**, 281–288 (1959).

[3] Nassif, M., Products and zeros of basic sets, Assiut (1958).

[4] Whittaker, J. M., Interpolatory Function Theory (Cambridge Press, 1935).

[5] Whittaker, J. M., Sur les Séries de Base de Polynomes Quelconques, Paris (1949)

(Paper 76B3&4–368)