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A Note on the Time Dependence of the Effective Axis and Angle of a Rotation*

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The time dependent rotation of one orthogonal coordinate system with respect to a fixed one is considered in the parametrization based on the effective axis and angle of the rotation, a parametrization which has recently been used to discuss the irreducible representations of the rotation group. The method of the intrinsic vector is used to derive the equations of motion for the instantaneous effective axis and angle. A new representation of the angular velocity is obtained in a rotating orthogonal coordinate system generated by a unit vector along the effective axis, and a new geometrical interpretation of the effective angle is given.

Key words: Angular velocity; effective axis and angle of rotation; intrinsic vector; kinematics of a rigid body; rotation group; spatial rotation.

1. Introduction

The most common parametrization of the rotation group is by means of the Euler angles [1, 2].[1] Recently, there has been some interest in the parametrization of this group by the direct use of the effective axis and angle of a rotation. Moses [3, 4] has calculated the irreducible representations in that parametrization, and [5] has computed the orthogonality relations between the matrices of any two such irreducible representations. Carmeli [6] has obtained the same results using a technique which differs from Moses', and which is originally due to Weyl. The outstanding feature of these results is their remarkable simplicity, compared to the corresponding results in the parametrization through the Eulerian angles.

The most familiar use of the Euler angles is of course in the description of the rotational motion of an orthogonal coordinate system. The instantaneous angular velocity is related to the time rate of change of the orientation of the rotating coordinate system by the well known equations [7]

$$\omega_1 = \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi \tag{1a}$$

$$\omega_2 = \frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\sin\theta\cos\phi \tag{1b}$$

$$\omega_3 = \frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta, \tag{1c}$$

where ω_1, ω_2, and ω_3 are the components of the angular velocity in a space-fixed coordinate system, and ϕ, θ, and ψ are the Euler angles which describe the instantaneous orientation of the rotating coordinate system in the spaced-fixed system. If $\boldsymbol{\omega}(t)$ is regarded as known, eqs (1) are the differential equations governing the time development of the orientation of the rotating system.

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[1] Figures in brackets indicate the literature references at the end of this paper.

With the increased use of the effective axis and angle to parametrize a rotation, it is of interest to examine the relation between the instantaneous angular velocity of a rotating coordinate system and the time development of its orientation when the orientation is specified by an effective axis and angle. Such an examination would give additional insight into this simple, yet rarely used parametrization of rotations. That examination is the purpose of this paper.

In section 2, the method of the intrinsic vector [8, 9] is used to derive expressions for the time rate of change of the effective axis and angle of rotation, induced by the angular velocity. A decomposition is then given of the angular velocity in an orthogonal set of vectors generated by a unit vector along the effective axis. This decomposition is the analog of eqs (1) in the parametrization based on the effective axis and angle. It is used to construct a new geometrical interpretation of the effective angle of rotation. The question of the extent to which the motion of the effective axis differs from the rigid rotation of an axis fixed in the rotating system, is briefly considered. It is shown that the motion of the effective axis fulfills the condition of rigid rotation only in the simple case when the rotating coordinate system rotates about a fixed axis.

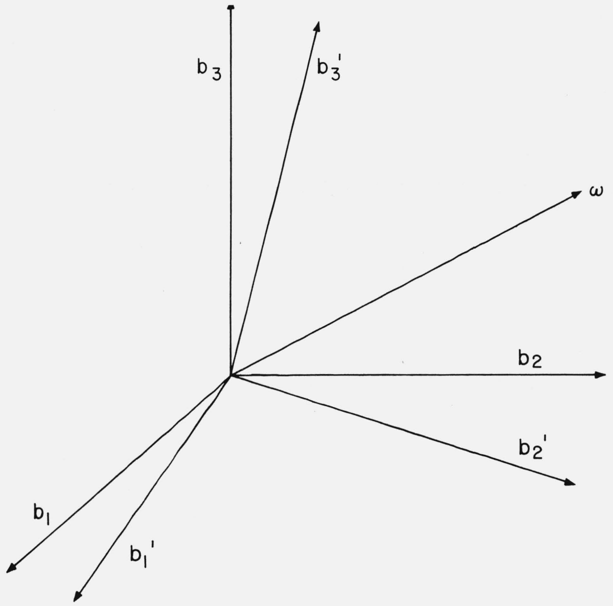

FIGURE 1. *Instantaneous orientation of a coordinate system rotating with respect to a fixed one.*

2. Time Development of the Orientation of a Rotating Coordinate System

We consider a pair of co-original orthogonal coordinate systems S and S', of the same handedness, specified, respectively, by the triads of unit vectors $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ and $(\mathbf{b}_1', \mathbf{b}_2', \mathbf{b}_3')$, and take the point of view that S' is rotating with respect to S. The situation is shown in figure 1, in which $\boldsymbol{\omega}$ represents the instantaneous angular velocity of S'. At any instant t the triads $\{\mathbf{b}_i\}$ and $\{\mathbf{b}_i'\}$ are linearly related by the orthogonal rotation dyadic $\mathsf{A}(t)$ according to the equation

$$\mathbf{b}_i'(t) = \mathbf{b}_i \cdot \mathsf{A}(t) = \tilde{\mathsf{A}}(t) \cdot \mathbf{b}_i , \tag{2}$$

where $\tilde{\mathsf{A}}$ is the dyadic conjugate to A. The matrix elements of A in the coordinate system S are given by

$$A_{ij} = \mathbf{b}_i \cdot \mathsf{A} \cdot \mathbf{b}_j = \mathbf{b}_i' \cdot \mathbf{b}_j . \tag{3}$$

The time derivative of $\tilde{\mathbf{A}}$ and the instantaneous angular velocity are related by

$$\frac{d\tilde{\mathbf{A}}}{dt}=\boldsymbol{\omega}\times\tilde{\mathbf{A}}=(\boldsymbol{\omega}\times\mathbf{I})\cdot\tilde{\mathbf{A}}, \tag{4}$$

in which $\mathbf{I}$ is the unit dyadic. Using the fact that the conjugate of $\boldsymbol{\omega}\times\mathbf{I}$ is $-\boldsymbol{\omega}\times\mathbf{I}$, and the orthogonality of $\mathbf{A}$,

$$\tilde{\mathbf{A}}\cdot\mathbf{A}=\mathbf{A}\cdot\tilde{\mathbf{A}}=\mathbf{I}, \tag{5}$$

we may write eq (4) as

$$\frac{d\mathbf{A}}{dt}=-\mathbf{A}\cdot(\boldsymbol{\omega}\times\mathbf{I}). \tag{6}$$

In matrix element form, this equation is

$$\frac{dA_{ij}}{dt}=A_{ik}\epsilon_{kjm}\omega_m, \tag{7}$$

where ϵ_{kjm} is the completely antisymmetric Levi-Civita symbol.

The effective axis and angle of rotation for time t are extracted from $\mathbf{A}(t)$ by computing the trace of $\mathbf{A}$, tr $\mathbf{A}$, and constructing the intrinsic vector [9],

$$V_i=\epsilon_{ijk}A_{jk}. \tag{8}$$

These quantities are related to the effective angle of rotation α, and a unit vector $\mathbf{n}$ along the effective axis of rotation by the formulas

$$\operatorname{tr}\mathbf{A}=1+2\cos\alpha, \tag{9}$$

$$\mathbf{V}=2\mathbf{n}\sin\alpha. \tag{10}$$

The handedness of the description of α with respect to $\mathbf{n}$ is fixed by eq (10) to agree with the common handedness of the coordinate systems S and S'.

By differentiating eq (8) with respect to t, using eqs (7) and (10) to evaluate dA_{jk}/dt and $d\mathbf{V}/dt$ respectively, and making use of the identity

$$\epsilon_{ijk}\epsilon_{msk}=\delta_{im}\delta_{js}-\delta_{is}\delta_{jm}, \tag{11}$$

we arrive at

$$2\frac{d\mathbf{n}}{dt}\sin\alpha+2\mathbf{n}\frac{d\alpha}{dt}\cos\alpha=\mathbf{B}\cdot\boldsymbol{\omega}, \tag{12}$$

where the dyadic $\mathbf{B}$ is given by

$$\mathbf{B}=(1+2\cos\alpha)\mathbf{I}-\tilde{\mathbf{A}}. \tag{13}$$

Equation (12) is the starting point for computing the equations of motion for $\mathbf{n}$ and α.

We first take the scalar product of eq (12) with $\mathbf{n}$. Since $\mathbf{n}$ is a unit vector, it is perpendicular to its time derivative. Furthermore, from Euler's theorem [10]

$$\mathbf{n}\cdot\tilde{\mathbf{A}}\equiv\mathbf{A}\cdot\mathbf{n}=\mathbf{n}. \tag{14}$$

The result of dotting eq (12) with $\mathbf{n}$ is therefore

$$2\frac{d\alpha}{dt}\cos\alpha=(1+2\cos\alpha)\,\boldsymbol{\omega}\cdot\mathbf{n}-\boldsymbol{\omega}\cdot\mathbf{n},$$

$$=2\boldsymbol{\omega}\cdot\mathbf{n}\cos\alpha.$$

Cancelling the common factor $2\cos\alpha$, we have

$$\frac{d\alpha}{dt}=\boldsymbol{\omega}\cdot\mathbf{n}. \tag{15}$$

Equation (15) states the compellingly simple, and intuitively almost obvious result that the instantaneous rate of change of the effective angle of rotation, is given by the component of the instantaneous angular velocity along the instantaneous effective axis of rotation.

We now insert the trigonometric representation of $\tilde{\mathbf{A}}$ [11],

$$\tilde{\mathbf{A}}=\mathbf{nn}+\mathbf{n}\times\mathbf{I}\sin\alpha-\mathbf{n}\times(\mathbf{n}\times\mathbf{I})\cos\alpha,$$

into eq (13) and use the result to evaluate $\mathbf{B}\cdot\boldsymbol{\omega}$ in eq (12). After replacing *dα/dt by* $\boldsymbol{\omega}\cdot\mathbf{n}$ and collecting similar terms we have

$$2\frac{d\mathbf{n}}{dt}\sin\alpha+\mathbf{n}\times\boldsymbol{\omega}\sin\alpha+\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\omega})\,(1+\cos\alpha)=0. \tag{17}$$

This may also be written

$$\frac{d\mathbf{n}}{dt}=-\frac{1}{2}\left[\mathbf{n}\times\boldsymbol{\omega}+\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\omega})\cot\frac{\alpha}{2}\right]. \tag{18}$$

There is no actual singularity at $\alpha=0$ in eq (18) since for $\alpha=0$, $\mathbf{n}$ and $\boldsymbol{\omega}$ are parallel.

Equation (15) and either of eqs (17) and (18) constitute the coupled equations of motion for the effective axis and angle in terms of the angular velocity. The inverse problem to that of obtaining these equations, is the problem of constructing the angular velocity from $\mathbf{n}$, α, and their time derivatives. The construction is most readily carried out by formally solving eq (12) for $\boldsymbol{\omega}$. This requires the dyadic $\mathbf{B}^{-1}$ which is inverse to $\mathbf{B}$. To test whether the inverse exists, we compute det $\mathbf{B}$, which is most easily done by considering the special form of $\tilde{\mathbf{A}}(t)$ when $\mathbf{n}(t)$ instantaneously lies along one of the coordinate axes, say, the $\mathbf{b}_3$ axis. The result of the computation is

$$\det\mathbf{B}=4\cos\alpha\,(\cos\alpha+1). \tag{19}$$

Equation (19) shows that $\mathbf{B}^{-1}$ does not formally exist when α is $\pi/2$, π, or $3\pi/2$. In constructing $\mathbf{B}^{-1}$, we shall therefore assume that α is not equal to any of these special angles. Nevertheless, the final result for $\boldsymbol{\omega}$ will be seen to be well defined for all α.

Using the trigonometric representation (16), and the identity

$$\mathbf{n}\times(\mathbf{n}\times\mathbf{I})\equiv\mathbf{nn}-\mathbf{I},$$

we may write $\mathbf{B}$ in the form

$$\mathbf{B}=2[v^2\mathbf{I}-u^2\mathbf{nn}-uv\mathbf{n}\times\mathbf{I}], \tag{20}$$

where

$$u=\sin\frac{\alpha}{2}, \tag{21a}$$

$$v=\cos\frac{\alpha}{2}. \tag{21b}$$

We then assume an expansion of $\mathbf{B}^{-1}$ of the form

$$\mathbf{B}^{-1}=P\mathbf{I}+Q\mathbf{nn}+R\mathbf{n}\times\mathbf{I}, \tag{22}$$

and compute P, Q, and R from the requirement

$$\mathbf{B}\cdot\mathbf{B}^{-1}=\mathbf{B}^{-1}\cdot\mathbf{B}=\mathbf{I}. \tag{23}$$

The results are

$$P=\frac{1}{2}, \tag{24a}$$

$$Q=\frac{u^2}{v^2-u^2}, \tag{24b}$$

$$R=\frac{1}{2}\frac{u}{v}. \tag{24c}$$

The expression for $\mathbf{B}^{-1}$ in terms of full angles is

$$\mathbf{B}^{-1}=\frac{1}{2}\left[\mathbf{I}+\frac{1-\cos\alpha}{\cos\alpha}\mathbf{nn}+\frac{\sin\alpha}{1+\cos\alpha}\mathbf{n}\times\mathbf{I}\right]. \tag{25}$$

If we now multiply eq (12) from the left by $\mathbf{B}^{-1}$, and note that $\mathbf{n}$ is perpendicular to its time derivative, and that $\mathbf{n}\times\mathbf{I}\cdot\mathbf{n}=0$, we arrive at

$$\boldsymbol{\omega}=\mathbf{n}\frac{d\alpha}{dt}+\frac{d\mathbf{n}}{dt}\sin\alpha+\mathbf{n}\times\frac{d\mathbf{n}}{dt}(1-\cos\alpha). \tag{26}$$

Equation (26) is the analog of eqs (1) in the parametrization of the rotation through $\mathbf{n}$ and α, and gives the expansion of $\boldsymbol{\omega}(t)$ in the instantaneous mutually orthogonal trio of vectors $\mathbf{n}$, $d\mathbf{n}/dt$, and $\mathbf{n}\times d\mathbf{n}/dt$.

An interesting geometrical interpretation of the angle α emerges from eq (26). To arrive at this interpretation, we first construct the unit vectors which are parallel to $d\mathbf{n}/dt$ and $\mathbf{n}\times d\mathbf{n}/dt$ respectively. The lengths of these vectors are equal and, from eqs (15) and (18), are given by

$$\left|\frac{d\mathbf{n}}{dt}\right|=\left|\mathbf{n}\times\frac{d\mathbf{n}}{dt}\right|=\frac{1}{2}\sqrt{\omega^2-\left(\frac{d\alpha}{dt}\right)^2}\,csc\,\frac{\alpha}{2}. \tag{27}$$

The quantity $[\omega^2 - (d\alpha/dt)^2]^{1/2}$ is the component of $\boldsymbol{\omega}$ perpendicular to $\mathbf{n}$. If β is the angle between $\boldsymbol{\omega}$ and $\mathbf{n}$, we may write

$$\sqrt{\omega^2 - \left(\frac{d\alpha}{dt}\right)^2} = \omega \sin \beta. \tag{28}$$

We now define three mutually orthogonal unit vectors $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ as follows:

$$\mathbf{q}_1 \equiv \frac{d\mathbf{n}/dt}{|d\mathbf{n}/dt|}, \tag{29a}$$

$$\mathbf{q}_2 \equiv \mathbf{n} \times \mathbf{q}_1, \tag{29b}$$

$$\mathbf{q}_3 \equiv \mathbf{n}. \tag{29c}$$

In terms of these vectors and the angle β in eq (28), we may rewrite eq (26) in the form

$$\boldsymbol{\omega} = \omega\left(\mathbf{q}_1 \sin \beta \cos \frac{\alpha}{2} + \mathbf{q}_2 \sin \beta \sin \frac{\alpha}{2} + \mathbf{q}_3 \cos \beta\right). \tag{30}$$

From eq (30) we see that the angle $\alpha/2$ is the azimuthal angle of $\boldsymbol{\omega}$ with respect to the plane of $\mathbf{q}_1$ and $\mathbf{q}_3$, that is, the plane of $\mathbf{n}$ and $d\mathbf{n}/dt$. This interpretation is illustrated in figure 2, which shows the polar representation of $\boldsymbol{\omega}$ in the coordinate system formed by $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$.

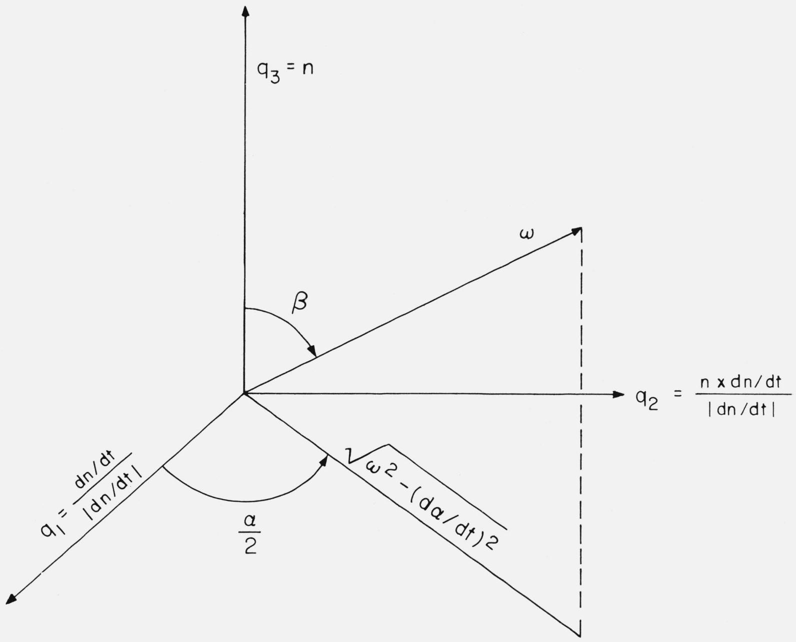

FIGURE 2. *Polar representation of the instantaneous angular velocity in a rotating coordinate system generated by a unit vector along the effective axis, and by the time derivative of that unit vector.*

The condition that the effective axis rotate rigidly with S' is

$$\frac{d\mathbf{n}}{dt} = \boldsymbol{\omega} \times \mathbf{n}. \tag{31}$$

From eq (18), it is clear that in general, the motion of $\mathbf{n}$ departs significantly from a rigid rotation. The condition (31) is so restrictive that the only class of motions of S' for which it is obeyed is that for which $\mathbf{n}$ is fixed in both S and S', with S' consequently rotating about the fixed direction of $\mathbf{n}$. This can be shown directly from eq (26), but is most easily shown from Euler's theorem. Differentiating eq (14), and making use of eq (6), we have

$$\mathsf{A}\cdot\left(\frac{d\mathbf{n}}{dt}-\boldsymbol{\omega}\times\mathbf{n}\right)=\frac{d\mathbf{n}}{dt}. \tag{32}$$

From eqs (31) and (32) we see that the condition that $\mathbf{n}$ rotate rigidly with S' leads to

$$\frac{d\mathbf{n}}{dt}=0. \tag{33}$$

Equation (26) then reduces to

$$\boldsymbol{\omega}=\mathbf{n}\frac{d\alpha}{dt}. \tag{34}$$

3. References

[1] Rose, M. E., Elementary Theory of Angular Momentum, Sec. 13 (John Wiley and Sons, Inc., New York, 1957).

[2] Murnaghan, Francis D., The Theory of Group Representations, Chap. 10, Sec. 1 (Johns Hopkins Press, Baltimore, 1938); (Dover Publications, Inc., New York, 1963).

[3] Moses, H. E., Irreducible Representations of the Rotation Group in Terms of Euler's Theorem. Nuovo Cimento **40A** (1965), 1120–1138.

[4] Moses, H. E., Irreducible Representations of the Rotation Group in Terms of the Axis and Angle of Rotation. Ann. Phys. (N.Y.) **37** (1966), 224–226.

[5] Moses, H. E., Irreducible Representations of the Rotation Group in Terms of the Axis and Angle of Rotation II. Orthogonality Relations Between Matrix Elements and Representations of Rotations in the Parameter Space. Ann. Phys. (N.Y.) **42** (1967), 343–346.

[6] Carmeli, M., Representations of the Three-Dimensional Rotation Group in Terms of Direction and Angle of Rotation. J. Math. Phys. **9** (1968), 1987–1992.

[7] Corben, H. C., and Stehle, P., Classical Mechanics, 2d ed., Sec. 49 (John Wiley and Sons, Inc., New York/London, 1960).

[8] Gelman, H., The Second Orthogonality Conditions in the Theory of Proper and Improper Rotations, I. Derivation of the Conditions and of Their Main Consequences. J. Res. Nat. Bur. Stand. (U.S.), 72B (Math. Sci.) No. 3 (July–Sept, 1968), 229–237.

[9] Gelman, H., The Second Orthogonality Conditions in the Theory of Proper and Improper Rotations, II. The Intrinsic Vector. J. Res. Nat. Bur. Stand. (U.S.), 73B (Math. Sci.), No. 2 (April–June, 1969), 125–138.

[10] Goldstein, H., Classical Mechanics, Sec. 4–6 (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1950).

[11] Page, L., Introduction to Theoretical Physics, 3d ed., Sec. 18 (D. Van Nostrand Company, Inc., New York, 1952).

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