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On Entire Functions of Exponential Type*

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Let f be an entire function and let $p \geq 1$ and

$$I(l, r) = \left\{ \int_0^{2\pi} | f^{(l)}(re^{i\theta}) |^p d\theta \right\}^{1/p}.$$

Let $C > 0$. If there exists a positive integer N such that for $k = 0, 1, \ldots, N$,

$$\sum_{j=0}^{N} \frac{I(k+j, r)}{j!} \geq C \sum_{j=N+1}^{\infty} \frac{I(k+j, r)}{j!},$$

for all sufficiently large r, then f is of exponential type not exceeding $\left\{2 \log \left(1 + \frac{1}{C}\right) + 1 + \log (2N)!\right\}$.

If this condition is replaced by related conditions, then also is of exponential type.

Key words: Bounded index; entire function; exponential type; maximum modulus.

1. Introduction

An entire function $f(z)$ is said to be of bounded index if and only if there exists a non-negative integer N (independent of z) such that

$$\max_{0 \leq j \leq N} \frac{| f^{(j)}(z) |}{j!} \geq \frac{| f^{(k)}(z) |}{k!} \tag{1.1}$$

for all k and all z, and the smallest such integer N is called the index of $f(z)$ ([1], [4], [5]).[1] It is known that a function of bounded index N is of exponential type not exceeding $N+1$ [6] but that a function of exponential type need not be of bounded index. In fact any entire function having zeros of arbitrarily large multiplicity is not of bounded index and there exist functions with simple zeros and of exponential type which are not of bounded index [8]. In a recent paper [2] Fred Gross considers interesting variations of condition (1.1) and proves the following

THEOREM A: *Let* f *be entire and* C *a positive constant. If there exists a positive integer* N *such that for* k = 0, 1, . . ., N, f *satisfies one of the following, for all* z *with* | z | *sufficiently large:*

(i)
$$\sum_{j=0}^{N} \frac{|f^{(k+j)}(z)|}{j!} > C \sum_{j=N+1}^{\infty} \frac{|f^{(k+j)}(z)|}{j!},$$

(ii)
$$\sum_{j=0}^{N} \frac{I(k+j, r)}{j!} > C \sum_{j=N+1}^{\infty} \frac{I(k+j, r)}{j!} \quad (\text{p } \textit{some positive integer}),$$

(iii)
$$\sum_{j=0}^{N} \frac{M(r, f^{(k+j)})}{j!} > C \sum_{j=N+1}^{\infty} \frac{M(r, f^{(k+j)})}{j!},$$

then f *is of exponential type.*

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[1] Figures in brackets indicate the literature references at the end of this paper.

Assuming the known properties of the maximum term and the central index of an entire function we obtain a bound on the exponential type T defined by

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r} = T.$$

Also we relax the hypothesis in each case. We prove

THEOREM 1: *Let* $p \geqslant 1$ *and* $C > 0$ *be two given constants. Let* $f(z)$ *be entire,* $z = re^{i\Theta}$, *and suppose that there exists a positive integer* N *(independent of* z*) such that for* $k = 0, 1, 2, \ldots, N$, *the following inequality*

$$\sum_{j=0}^{N} \frac{I(k+j, r)}{j!} \geqslant C \sum_{j=N+1}^{\infty} \frac{I(k+j, r)}{j!} \tag{1.2}$$

holds for all z *with* $|z|$ *sufficiently large. Then* $f(z)$ *is of exponential type and*

$$T \leqslant 1 + 2\log\left(1 + \frac{1}{C}\right) + \log (2N)!.$$

THEOREM 2: *Let* $f(z)$ *be entire and* C *a positive constant. Suppose that there exist two non-negative integers* k *and* N *(independent of* z*) such that the following inequality*

$$\sum_{j=0}^{N} \frac{|f^{(k+j)}(z)|}{j!} \geqslant C \sum_{j=N+1}^{\infty} \frac{|f^{(k+j)}(z)|}{j!} \tag{1.3}$$

holds for all z *with* $|z|$ *sufficiently large. Then* $f(z)$ *is of exponential type and*

$$T \leqslant \max\left\{N, \min_{1\leqslant j\leqslant N} \left(\frac{(N+j)!(N+1)}{(N!)C}\right)^{1/j}\right\}.$$

THEOREM 3: *Let* $f(z)$ *be entire and* C *a positive constant. Write* $M(r, f^{(1)}) = \max|f^{(1)}(z)|$. *Suppose that there exist two non-negative integers* k *and* N *(independent of* z*) such that the following inequality*

$$\sum_{j=0}^{N} \frac{M(r, f^{(k+j)})}{j!} \geqslant C \sum_{j=N+1}^{\infty} \frac{M(r, f^{(k+j)})}{j!} \tag{1.4}$$

holds for all sufficiently large r. *Then* $f(z)$ *is of exponential type and*

$$T \leqslant \left(\frac{(2N+1)!}{C(N!)}\right)^{1/(N+1)}$$

2. Proof of Theorem 1.

We have for $r > r_0$

$$\sum_{j=0}^{N} \frac{I(k+j, r)}{j!} = \sum_{j=0}^{N} \frac{I(k+j, r)}{(k+j)!} \frac{(k+j)!}{j!}$$

$$\leqslant (2N)!\sum_{j=0}^{2N}\frac{I(j,r)}{j!}$$

$$=(2N)!\left\{\left(\sum_{j=0}^{N}+\sum_{j=N+1}^{2N}\right)\frac{I(j,r)}{j!}\right.$$

$$<(2N)!\left(1+\frac{1}{C}\right)\sum_{j=0}^{N}\frac{I(j,r)}{j!}\quad. \tag{2.1}$$

Here we have used the hypothesis with $k=0$ and $r>r_0$ to obtain the last inequality. From the Taylor expansion

$$f^{(k)}(a+h)=\sum_{j=0}^{\infty}\frac{f^{(k+j)}(a)}{j!}h^j$$

we have

$$|f^{(k)}(re^{i\theta})|^p=\left|\sum_{j=0}^{\infty}\frac{f^{(k+j)}((r-1)e^{i\theta})}{j!}e^{ij\theta}\right|^p$$

and so

$$I(k,r)\leqslant\left\{\int_0^{2\pi}\left\{\sum_{j=0}^{\infty}\frac{|f^{(k+j)}((r-1)e^{i\theta})|}{j!}\right\}^p d\theta\right\}^{1/p}\quad.$$

We now use Minkowski inequality [3, p. 148] to obtain

$$I(k,r)\leqslant\sum_{j=0}^{\infty}\frac{I(k+j,r-1)}{j!}.$$

Using the hypothesis and (2.1), we have for $r>r_1>1+r_0$,

$$I(k,r)\leqslant\left(1+\frac{1}{C}\right)\sum_{j=0}^{N}\frac{I(k+j,r-1)}{j!}\quad.$$

$$\leqslant\left(1+\frac{1}{C}\right)^2(2N)!\sum_{j=0}^{N}\frac{I(j,r-1)}{j!}\quad.$$

Hence

$$\sum_{k=0}^{N}\frac{I(k,r)}{k!}\leqslant e\left(1+\frac{1}{C}\right)^2(2N)!\sum_{j=0}^{N}\frac{I(j,r-1)}{j!}.$$

Write $\lambda=e\left(1+\frac{1}{C}\right)^2(2N)!$ and $\sum_{k=0}^{N}\frac{I(k,r)}{k!}=\xi(r)$. Then $\xi(r)\leqslant\lambda\,\xi(r-1)$ and so we get, for $r>r_1$,

$$\xi(r)<C_1\lambda^r, \tag{2.2}$$

where $C_1 = C_1(N, p, r_1)$ is a constant. Write now $f(z) = \sum_0^\infty a_n z^n$, $\mu(r) = \max_n |a_n| r^n$. Then

$$|a_n| r^n \leqslant \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta$$

$$\leqslant \frac{1}{2\pi} \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p} \left\{ \int_0^{2\pi} d\theta \right\}^{1/q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence (2.2) gives

$$\mu(r) < (2\pi)^{-1/p} C_1 \lambda^r. \tag{2.3}$$

Now [10; 32–34]

$$\limsup_{r \to \infty} \frac{\log \log \mu(r)}{\log r} = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = \rho \text{ (say)},$$

and if $\rho < \infty$, then

$$\log \mu(r) \sim \log M(r).$$

These two relations along with (2.3) imply that $f(z)$ is of exponential type and $T \leqslant \log \lambda$. The proof is complete.

3. Proof of Theorem 2

The proof of this theorem is essentially the same as that of Theorem 1 of [6]. We therefore outline the proof. Write $f^{(k)}(z) = F(z)$. Then

$$\sum_{j=0}^{N} \frac{|F^{(j)}(z)|}{j!} \geqslant C \sum_{j=N+1}^{\infty} \frac{|F^{(j)}(z)|}{j!}.$$

Hence for $p \geqslant 1$

$$\sum_{j=0}^{N} \frac{M(r, F^{(j)})}{j!} \geqslant C \frac{M(r, F^{(N+p)})}{(N+p)!}.$$

Let

$$\limsup_{r \to \infty} \frac{\nu(r, f)}{r} = a.$$

Then (cf: [6]) there exists a sequence $r_n \uparrow \infty$ such that for $r = r_n$, $(n > n_0)$,

$$\frac{M(r, F^{(j)})}{M(r, F^{(q)})} \sim \left(\frac{\nu(r, F)}{r} \right)^{j-q} \geqslant b^{j-q}, \qquad 0 \leqslant q \leqslant j,\ j = 1, 2, \ldots, 2N,\ b < a.$$

Hence if $b_1 < b$, we get for $1 \leqslant j \leqslant N$, *and* $r = r_n$,

$$CM(r, F^{(N+j)}) < (N+j)!\left\{\frac{M(r, F^{(N+j)})}{b_1^{N+j}} + \ldots + \frac{M(r, F^{(N+j)})}{N!\, b_1^j}\right\},$$

and so we have for $1 \leqslant j \leqslant N$,

$$\frac{Cb_1^{N+j}}{(N+j)!} \leqslant 1 + \frac{b_1}{1!} + \ldots \frac{b_1^N}{N!}. \tag{3.1}$$

If $b_1 < N$ then we get $a \leqslant N$ and so $T \leqslant N$. If $b_1 \geqslant N$ then we have from (3.1)

$$\frac{Cb_1^{N+j}}{(N+j)!} \leqslant (N+1)\frac{b_1^N}{N!}$$

and so

$$T \leqslant \min_{1 \leqslant j \leqslant N}\left\{\frac{(N+j)!(N+1)}{N!C}\right\}^{1/j}.$$

The proof is complete.

The proof of Theorem 3 is similar and omitted.

4. Remarks and Examples

(a) There are functions satisfying all hypotheses (i)–(iii) of Theorem A. Take, for instance, $f(z) = e^{az}$, $a > 0$. Gross [2] has proved that a periodic function of exponential type satisfies (ii) and (iii) of Theorem A for sufficiently large r and $i = 0, 1, \ldots, N$.

(b) Let $f(z)$ be entire and suppose that it satisfies the differential equation

$$P_0(z)f^{(k)}(z) + P_1(z)f^{(k-1)}(z) + \ldots + P_k(z)f(z) = Q(z), \tag{4.1}$$

where $P_J(z)$, $J = 0, 1, \ldots, k$, $Q(z)$ are polynomials and $P_0(z)$ $(\not\equiv 0)$ is of degree not less than that of any $P_J(z)$. Then $f(z)$ is of bounded index [7]. Furthermore $f(z)$ satisfies the hypothesis of Theorem 2, that is, inequality (1.3) with $k = 0$ and for all z. For let $\alpha = 1 + C$ and $F(z) = f(\alpha z)$. Then $F(z)$ also satisfies a differential equation of the form (4.1). Hence $F(z)$ is of bounded index N say, and for all k and all z

$$\frac{|F^{(k)}(z)|}{k!} \leqslant \max_{0 \leqslant s \leqslant N} \frac{|F^{(s)}(z)|}{s!}.$$

This implies that

$$\frac{|f^{(k)}(z)|}{k!} \leqslant \alpha^{N-k} \max_{0 \leqslant s \leqslant N} \frac{|f^{(s)}(z)|}{s!}.$$

Hence for $k = N+1, N+2, \ldots$,

$$\frac{M(r, f^{(k)})}{k!} \leqslant \alpha^{N-k}\Omega(r)$$

where

$$\Omega(r) = \max_{0 \leqslant s \leqslant N}\left\{\frac{M(r, f^{(s)})}{s!}\right\}.$$

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This gives

$$\sum_{k=N+1}^{\infty} \frac{M(r, f^{(k)})}{k!} \leqslant \frac{1}{C}\Omega(r) < \frac{1}{C}\sum_{0}^{N} \frac{M(r, f^{(s)})}{s!}.$$

Similarly the entire solution $f(z)$ of (4.1) satisfies the hypothesis of Theorem 3 with $k=0$ and for all z. It also satisfies the hypothesis of Theorem 1, that is, given $p \geqslant 1$ and $C > 0$, there exists a positive integer N such that for $k=0, 1, 2, \ldots, N$ and for all $r=|z|$, (1.2) holds. To prove this we let $\alpha = 1+C$ and denote by $M \equiv M_0, M_1, \ldots, M_M$, the indices of the functions $f(\alpha z) \equiv F(z)$, $F'(z), \ldots, F^{(M)}(z)$ and let

$$N = \max_{0 \leqslant J \leqslant M} M_J.$$

Then the inequality (1.2) holds for $k=0, 1, \ldots, N$. We omit the details.

(c) We now show, in Example 1, that there exist entire functions which satisfy the hypothesis of Theorem 3 (with $k=0$, $N=0$ and given C) but which do not satisfy the hypothesis of Theorem 2. In Example 2 we give a function which does not satisfy the hypothesis (i) of Theorem A but which satisfies the hypothesis of corresponding Theorem 2.

Example 1: Let $C > 0$ be a given constant. Let

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{na^{n+1}}\right)^n$$

where we choose $a \geqslant 2$ such that $\dfrac{1}{a(a-1)} \leqslant \log(1+C)$. Then $f(z)$ is entire and satisfies the conditions of Theorem 3, with $k=0$, $N=0$ and given C. Since f has zeros of arbitrarily large multiplicity, the hypothesis of Theorem 2 cannot be satisfied.

Example 2: Let $a_1 = k_1 = 10$; $k_{j+1} = k_j^2$,

$$a_{j+1} = k_{j+1} \exp\left\{\left(\log \frac{3}{2}\right) k_j\right\}, \; j \geqslant 1$$

$$f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right)^{k_j}, \; F(z) = f(z) - 1.$$

Then $F(z)$ is an entire function of bounded index [9] and $F'(z)$ has zeros of arbitrarily large multiplicity. Let $C > 0$, $a = 1+C$, $g(z) = F\left(\dfrac{z}{a}\right)$. Since $F(z)$ has an infinity of zeros, the index N of $F(z)$ is a positive integer and we have for all z and k,

$$\max_{0 \leqslant j \leqslant N} \frac{|F^{(j)}(z)|}{j!} \geqslant \frac{|F^{(k)}(z)|}{k!}.$$

Hence for all z

$$\sum_{j=N+1}^{\infty} \frac{|g^{(j)}(z)|}{j!} = \sum_{j=N+1}^{\infty} \frac{|F^{(j)}(z/a)|}{a^j j!}$$

$$\leqslant \sum_{j=N+1}^{\infty} \frac{1}{a^j} \max_{0 \leqslant k \leqslant N} \frac{|F^{(k)}(z/a)|}{k!}$$

$$= \frac{a}{a^{N+1}(a-1)} \max_{0 \leqslant k \leqslant N} \frac{a^k |g^k(z)|}{k!}$$

$$< \frac{a^{N+1}}{a^{N+1}(a-1)} \sum_{k=0}^{N} \frac{|g^{(k)}(z)|}{k!}$$

$$= \frac{1}{C} \sum_{k=0}^{N} \frac{|g^{(k)}(z)|}{k!}.$$

This shows that $g(z)$ satisfies the hypothesis of Theorem 2 with $k=0$. Further $g'(z)$ has zeros of arbitrarily large multiplicity and so $g(z)$ cannot satisfy the hypothesis (i) (when $k=1$) of Theorem A.

5. References

[1] Gross, F., Entire functions of bounded index, Proc. Amer. Math. Soc. **18** (1967), 974–980.

[2] Gross, F., Entire functions of exponential type, J. Res. Nat. Bur. Stand. (U.S.), 72B (Math. Sci.), No. 1, 55–59 (Jan.–Feb. 1970).

[3] Hardy, G. H., Littlewood, J. E., Polya, G., Inequalities (Camb. Univ. press 1964).

[4] Lepson, B., Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index, Lecture Notes. Summer Institute on Entire Functions, Univ. of California, La Jolla (1966).

[5] Lepson, B., Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index, Proc. Symp. in Pure Math. **XI,** 298–308 (1968).

[6] Shah, S. M., Entire functions of bounded index, Proc. Amer. Math. Soc. **19,** 1017–1022 (1968).

[7] Shah, S. M., Entire functions satisfying a linear differential equation, J. of Math. and Mech. **18,** 131–136 (1968).

[8] Shah, S. M., Entire functions of unbounded index and having simple zeros, Math. Zeit., **118,** 193–196 (1970).

[9] Shah, S. M., On entire functions of bounded index whose derivatives are of unbounded index, Jour. Lond. Math. Soc. (2) 4, 127–139 (1971).

[10] Valiron, G., Lectures on the general theory of integral functions (Chelsea, New York 1949).

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