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Eigenvalues of Sums of Hermitian Matrices. III.*

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Two classes of nonlinear inequalities for the eigenvalues of sums of Hermitian matrices are obtained. These nonlinear inequalities are shown to follow from linear inequalities established in parts I and II of this series. A new inequality for the singular values of matrix products is also obtained.

Keywords: Eigenvalues; linear inequalities; singular values.

1. Introduction.

Let $C=A+B$ where C, A, B are Hermitian matrices. In recent years a number of inequalities have been established linking the eigenvalues of C, A, B. For the most part these inequalities are linear, but a number of nonlinear inequalities are also known which involve convex or concave functions of the eigenvalues. Generally speaking, the methods used to derive the linear inequalities work equally well in deriving these nonlinear inequalities. These methods are based on the extremal properties of eigenvalues or on induction and the Cauchy interlacing inequalities. A defect of the use of these methods when deriving nonlinear inequalities is that they do not clarify whether the nonlinear inequalities are consequences of the linear ones. There is, however, a third method which derives nonlinear inequalities from linear ones. It is the purpose of this paper to use this third method to obtain several new families of nonlinear inequalities for the eigenvalues of matrix sums. It will turn out that our new inequalities include as a special case a generalized and sharpened version of an inequality recently proved by Marcus [2].[1] We shall also derive a previously unnoticed inequality for the singular values of a matrix product.

2. The Main Results for Sums

THEOREM 1: *Let integers* $i_1, \ldots, i_m, j_1, \ldots, j_m$ *satisfy*

$$1 \leq i_1 < \ldots < i_m \leq n, \qquad 1 \leq j_1 < \ldots < j_m \leq n, \qquad i_m + j_m - m \leq n.$$

Set

$$k_s = i_s + j_s - s \quad \textit{for} \quad s = 1, \ldots, m.$$

Let $0 \leq \theta \leq 1$, $\theta + \varphi = 1$ *and let* A, B, $C = \theta A + \varphi B$ *be n-square Hermitian matrices with eigenvalues*

$$\alpha_1 \geq \ldots \geq \alpha_n, \beta_1 \geq \ldots \geq \beta_n, \gamma_1 \geq \ldots \geq \gamma_n, \tag{1}$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

respectively. Let $f(x_1, \ldots, x_m)$ *be a real-valued function, convex, symmetric, and nondecreasing in each* x_t for $(x_1, \ldots, x_m)$ *lying in the* m-dimensional *hypercube* $\mathscr{I} \times \ldots \times \mathscr{I}$, *where* $\mathscr{I}$ *is the interval*

$$\mathscr{I} = [min\ (\alpha_n, \beta_n), max\ (\alpha_1, \beta_1)].$$

Then

$$f(\gamma_{k_1} \ldots, \gamma_{k_1}) \leqslant \theta f(\alpha_{i_1}, \ldots, \alpha_{i_m}) + \varphi f(\beta_{j_1}, \ldots \ldots \beta_{j_m}). \tag{2}$$

PROOF: The inequality

$$\sum_{s=1}^{m} \gamma_{k_s} \leqslant \theta \sum_{s=1}^{m} \alpha_{i_s} + \varphi \sum_{s=1}^{m} \beta_{j_s} \tag{3}$$

is known [7, Theorem 1]. Applying this for r in place of m, we have

$$\sum_{s=1}^{r} \gamma_{k_s} \leqslant \theta \sum_{s=1}^{r} \alpha_{i_s} + \varphi \sum_{s=1}^{r} \beta_{j_s}, \qquad r = 1, \ldots, m.$$

If we let

$$c_s = \gamma_{k_s}, d_s = \theta \alpha_{i_s} + \varphi \beta_{j_s}, \qquad s = 1, \ldots, m,$$

then $c_1 \geqslant \ldots \geqslant c_m$, $d_1 \geqslant \ldots \geqslant d_m$, and $c_1 + \ldots + c_r \leqslant d_1 + \ldots + d_r$ for $1 \leqslant r \leqslant m$. These conditions are known [5, Theorem 15] to imply that $f(c_1, \ldots, c_m) \leqslant f(d_1, \ldots, d_m)$. From this (2) is immediate.

This proof uses only standard techniques. Rather more interesting and surprising is that these same techniques can be used to establish the inequalities of the next theorem.

THEOREM 2: *Let* $C = \theta A + \varphi B$ *and* f *be as in Theorem 1. Let* $Z_1, \ldots, Z_{n-m}, W_1, \ldots, W_{n-m}$ *be integers satisfying*

$$Z_1 \leqslant \ldots \leqslant Z_{n-m} \leqslant m, \qquad W_1 \leqslant \ldots \leqslant W_{n-m} \leqslant m, \qquad Z_1 + W_1 \geqslant m.$$

Let $\delta_x(y)$ *be a jump function;* $\delta_x(y) = 1$ *if* $y > x$ *and* $\delta_x(y) = 0$ *if* $y \leqslant x$. *Let*

$$I_s = s + \delta_{Z_1}(s) + \ldots + \delta_{Z_{n-m}}(s),$$

$$J_s = s + \delta_{W_1}(s) + \ldots + \delta_{W_{n-m}}(s),$$

$$K_s = s + \delta_{Z_1+W_1-m}(s) + \ldots + \delta_{Z_{n-m}+W_{n-m}-m}(s), \qquad s = 1, \ldots, m.$$

Then

$$f(\gamma_{K1}, \ldots, \gamma_{Km}) \leqslant \theta f(\alpha_{I1}, \ldots, \alpha_{Im}) + \phi f(\beta_{J1}, \ldots, \beta_{Jm}). \tag{4}$$

PROOF: The following inequality is known:

$$\sum_{s=1}^{m} \gamma_{K_s} \leqslant \theta \sum_{s=1}^{m} \alpha_{I_s} + \phi \sum_{s=1}^{m} \beta_{J_s}. \tag{5}$$

This inequality may be obtained from Theorem 4 of [8] by using $\delta_x^*(y) = \delta_{x-1}(y)$. In order to prove (4) we need to show that

$$\sum_{s=1}^{r} \gamma_{K_s} \leqslant \theta \sum_{s=1}^{r} \alpha_{I_s} + \phi \sum_{s=1}^{r} \beta_{J_s}. \tag{6}$$

holds for each $r=1, \ldots, m$. For if this is so then the proof of Theorem 1 applies here also. Define nonnegative integers u and v by

$$Z_1 \leqslant \ldots \leqslant Z_u \leqslant r < Z_{u+1} \leqslant \ldots \leqslant Z_{n-m},$$

$$W_1 \leqslant \ldots \leqslant W_v \leqslant r < W_{v+1} \leqslant \ldots \leqslant W_{n-m}.$$

Without loss of generality let $u \leqslant v$. Define $\xi_1 \leqslant \ldots \leqslant \xi_{n-r}$, $\omega_1 \leqslant \ldots \leqslant \omega_{n-r}$ by

$$\begin{gathered} \xi_1 = Z_1-(m-r), \ldots, \xi_u = Z_u-(m-r),\ \xi_{u+1}=r, \ldots, \xi_{n-r}=r, \\ \omega_1 = W_1, \ldots, \omega_v = W_v,\ \omega_{v+1}=r, \ldots, \omega_{n-r}=r. \end{gathered} \tag{7}$$

Then also $\xi_1+\omega_1 \geqslant r$. Set

$$P_s = s+\delta_{\xi_1}(s)+\ldots+\delta_{\xi_{n-r}}(s),$$

$$Q_s = s+\delta_{\omega_1}(s)+\ldots+\delta_{\omega_{n-r}}(s),$$

$$R_s = s+\delta_{\xi_1+\omega_1-r}(s)+\ldots+\delta_{\xi_{n-r}+\omega_{n-r}-r}(s), \qquad s=1, \ldots, r.$$

Writing down the inequality (5) with r in place of m we have

$$\sum_{s=1}^{r} \gamma_{R_s} \leqslant \theta \sum_{s=1}^{r} \alpha_{P_s} + \phi \sum_{s=1}^{r} \beta_{Q_s}. \tag{8}$$

Now for $s \leqslant r$ we have

$$P_s = s+\sum_{\rho=1}^{n-r} \delta_{\xi_\rho}(s) = s+\sum_{\rho=1}^{u} \delta_{\xi_\rho}(s) \geqslant s+\sum_{\rho=1}^{u} \delta_{Z_\rho}(s) = I_s,$$

because

$$\delta_{\xi_\rho}(s)=0 \text{ for } s \leqslant r \qquad \text{and} \qquad \rho > u;$$

$$\delta_{\xi_\rho}(s) = \delta_{Z_\rho-(m-r)}(s) \geqslant \delta_{Z_\rho}(s) \qquad \text{for} \qquad \rho \leqslant u.$$

For $s \leqslant r$ we have

$$Q_s = s+\sum_{\rho=1}^{n-r} \delta_{\omega_\rho}(s) = s+\sum_{\rho=1}^{v} \delta_{\omega_\rho}(s) = s+\sum_{\rho=1}^{v} \delta_{W_\rho}(s) = s+\sum_{\rho=1}^{n-m} \delta_{W_\rho}(s) = J_s,$$

since

$$\delta_{\omega_\rho}(s) = 0 = \delta_{W_\rho}(s) \qquad \text{for} \qquad \rho > v.$$

Furthermore, for $s \leqslant r$ (using $Z_\rho+W_\rho-m=\xi_\rho+\omega_\rho-r$ for $\rho \leqslant u$ *and* $Z_\rho-m+W_\rho \leqslant \omega_\rho$ for all ρ) we have

$$K_s \geqslant s+\sum_{\rho=1}^{u} \delta_{Z_\rho+W_\rho-m}(s) + \sum_{\rho=u+1}^{v} \delta_{Z_\rho+W_\rho-m}(s) \geqslant s+\sum_{\rho=1}^{u} \delta_{\xi_\rho+\omega_\rho-r}(s) + \sum_{\rho=u+1}^{v} \delta_{\omega_\rho}(s)$$

$$= s+\sum_{\rho=1}^{u} \delta_{\xi_\rho+\omega_\rho-r}(s) + \sum_{\rho=u+1}^{n-r} \delta_{\xi_\rho+\omega_\rho-r}(s) = R_s.$$

Thus, for $s=1, \ldots, r$, we have $P_s \geqslant I_s$, $Q_s = J_s$, $K_s \geqslant R_s$. From this it follows that (6) is a consequence of (8). This completes the proof.

REMARK 1: In fact we have proved that if $f(x_1, \ldots, x_r)$ is convex, symmetric, and nondecreasing then

$$f(\gamma_{K_1}, \ldots, \gamma_{K_r}) \leqslant \theta f(\alpha_{I_1}, \ldots, \alpha_{I_r}) + \varphi f(\beta_{J_1}, \ldots, \beta_{J_r}) \qquad \text{for} \qquad r \leqslant m. \tag{9}$$

That the inequality (9) for $r < m$ is valid is a new result even when f is linear.

As a consequence of theorems 1 and 2, we obtain theorems 3 and 4.

THEOREM 3: *Let* A, B, C *be as in theorem 1. Let* $g(x_1, \ldots, x_m)$ *be a real-valued function, symmetric, convex, and nonincreasing in each variable when* $(x_1, \ldots, x_m) \in \mathfrak{I} \times \ldots \times \mathfrak{I}$. *Let* $i'_1, \ldots, i'_m, j'_1, \ldots, j'_m, k'_1, \ldots, k'_m$ *be integers with*

$$1 \leqslant i'_1 < \ldots < i'_m \leqslant n, \qquad 1 \leqslant j'_1 < \ldots < j'_m \leqslant n, \qquad i'_1 + j'_1 > n+1-m,$$

$$k'_s = i'_s + j'_s - s + m - n, \qquad 1 \leqslant s \leqslant m.$$

Then

$$g(\gamma_{k'_1}, \ldots, {}_{k'_m}) \leqslant \theta g(\alpha_{i'_1}, \ldots, \alpha_{i'_m}) + \varphi g(\beta_{j'_1}, \ldots, \beta_{j'_m}).$$

THEOREM 4: *Let* A, B, C, g *be as in theorem 3. Let integers* $Z'_1, \ldots, Z'_{n-m}, W'_1, \ldots, W'_{n-m}$ *satisfy*

$$0 \leqslant Z'_1 \leqslant \ldots \leqslant Z'_{n-m}, \qquad 0 \leqslant W'_1 \leqslant \ldots \leqslant W'_{n-m}, \qquad Z'_{n-m} + W'_{n-m} \leqslant m.$$

Let

$$I'_s = s + \sum_{\rho=1}^{n-m} \delta_{Z'_\rho}(s), \qquad J'_s = s + \sum_{\rho=1}^{n-m} \delta_{W'_\rho}(s),$$

$$K'_s = s + \sum_{\rho-1}^{n-m} \delta_{Z'_\rho} + {}_{W'_\rho}(s), \qquad s = 1, \ldots, m.$$

Then

$$g(\gamma_{K'_1}, \ldots, \gamma_{K'_m}) \leqslant \theta g(\alpha_{I'_1}, \ldots, \alpha_{I'_m}) + \varphi g(\beta_{J'_1}, \ldots, \beta_{J'_m}).$$

PROOF: Let $f(x_1, \ldots, x_m) = g(-x_1, \ldots, -x_m)$ and apply theorems 1 and 2 to $-C = \theta(-A) + \varphi(-B)$. This yields theorem 3 in a straightforward manner. To obtain theorem 4 take $Z_s = m - Z'_{m+1-s}$ and use $1 - \delta_z(m+1-s) = \delta_{m-z}(s)$.

From theorems 1–4 we deduce

$$F(\gamma_{k_1}, \ldots, \gamma_{k_m}) \geqslant \theta F(\alpha_{i_1}, \ldots, \alpha_{i_m}) + \varphi F(\beta_{j_1}, \ldots, \beta_{j_m})$$

$$F(\gamma_{K_1}, \ldots, \gamma_{K_m}) \geqslant \theta F(\alpha_{I_1}, \ldots, \alpha_{I_m}) + \varphi F(\beta_{J_1}, \ldots, \beta_{J_m})$$

$$G(\gamma_{k_1'}, \ldots, \gamma_{k_{m'}}) \geqslant \theta G(\alpha_{i_1'}, \ldots, \alpha_{i_{m'}}) + \varphi G(\beta_{J_1'}, \ldots, \beta_{j_{m'}})$$

$$G(\gamma_{K_1'}, \ldots, \gamma_{K_{m'}}) \geqslant \theta G(\alpha_{I_1'}, \ldots, \alpha_{I_{m'}}) + \varphi G(\beta_{J_1'}, \ldots, \beta_{J_{m'}})$$

where F is symmetric, concave, and nonincreasing on $\mathscr{I}\times\ldots\times\mathscr{I}$, and G is symmetric, concave, and nondecreasing on $\mathscr{I}\times\ldots\times\mathscr{I}$. One merely applies Theorems 1–4 to $f=-F$ and $g=-G$.

REMARK 2: Let A, B be positive definite and set $C=A+B$. Let $g(x_1,\ldots,x_m)=x_1^{-1}+\ldots+x_m^{-1}$. Writing $C=\frac{1}{2}(2A)+\frac{1}{2}(2B)$, we obtain from theorem 3 the inequality

$$4\sum_{s=1}^{m}\gamma_{k_{s'}}{}^{-1}\leq\sum_{s=1}^{m}\alpha_{i_{s'}}{}^{-1}+\sum_{s=1}^{m}\beta_{j_{s'}}{}^{-1}. \tag{10}$$

In particular if we take $i_s'=s$ and $j_s'=n-m+s$, for $s=1,\ldots,m$, then (10) becomes

$$4\sum_{s=1}^{m}\gamma_s^{-1}\leq\sum_{s=1}^{m}\alpha_s^{-1}+\sum_{s=1}^{m}\beta_{n-m+s}^{-1}. \tag{11}$$

If in theorem 4, we set $Z_1'=\ldots=Z_{n-m}'=m$, $W_1'=\ldots=W_{n-m}'=0$ and use this same g we obtain (11). The inequality (11) is clearly sharper than the inequality

$$2\sum_{s=1}^{m}\gamma_s^{-1}\leq\sum_{s=1}^{m}\alpha_s^{-1}+\sum_{s=1}^{m}\beta_{n-m+s}^{-1}. \tag{12}$$

The inequality (12) was proved in [2] under the additional hypothesis $m\leq n/2$. Thus theorems 3 and 4 both substantially generalize and sharpen theorem 2 of [2].

By further specializations of theorems 1 and 2 many additional nonlinear inequalities may be written down. In particular many of the results in [3, chap. 2], [4], [10, p. 110] may be generalized.

REMARK 3: If in theorems 1 and 2, we let A, B, $C=\theta A+\phi B$ be not necessarily Hermitian matrices and we let the numbers (1) denote the singular values of A, B, C respectively, then the conclusions of theorems 1 and 2 remain valid. This is because the inequalities (3) and (5) are known to be valid for the singular values of matrix sums. By this type of device many of the singular value inequalities in [1, chapter 2] and [4] may be be substantially generalized.

REMARK 4: The results of theorems 1 and 2 are used in [6].

3. A new inequality for the singular values of a matrix product.

THEOREM 5: *Let* A, B, C=AB *be a product of not necessarily Hermitian matrices. Let (1) deonote the singular values of* A, B, C *respectively. Let* $\mathrm{I_s}$, $\mathrm{J_s}$, $\mathrm{K_s}$ *denote the integers defined in theorem 2. Then*

$$\sum_{s=1}^{m}\gamma_{K_s}\leq\sum_{s=1}^{m}\alpha_{I_s}\beta_{J_s}. \tag{13}$$

PROOF: The inequality

$$\prod_{s=1}^{m}\gamma_{K_s}\leq\prod_{s=1}^{m}\alpha_{I_s}\beta_{J_s}$$

is known [9]. As in the proof of theorem 2 it implies

$$\prod_{s=1}^{r}\gamma_{K_s}\leq\prod_{s=1}^{r}\alpha_{I_s}\beta_{J_s}$$

for each $r=1, \ldots, m$. *If* we set

$$c_s = \log \gamma_{K_s}, \qquad d_s = \log \alpha_{I_s}\beta_{J_s}$$

then $c_1+\ldots+c_r \leq d_1+\ldots+d_r$ for $r=1, \ldots, m$. The function $f(x_1, \ldots, x_m) = \exp x_1 + \ldots + \exp x_m$ is convex and increasing and hence $\exp c_1 + \ldots + \exp c_m \leq \exp d_1 + \ldots + \exp d_m$. This immediately yields (13). The inequality

$$\sum_{s=1}^{m} \gamma_{k_s} \leq \sum_{s=1}^{m} \alpha_{i_s}\beta_{j_s}$$

is also true and was proved by the same method in [9].

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