Linear Transformations on Matrices*

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Let $K$ be a field and let $M_n(K)$ denote the vector space of all $n \times n$ matrices over $K$. Suppose $I(X)$ is an invariant defined on a subset $\mathfrak{A}$ of $M_n(K)$. This paper surveys certain results concerning the following problem. Describe the set $\mathcal{L}(I, \mathfrak{A})$ of all linear transformations $T: \mathfrak{A} \rightarrow \mathfrak{A}$ that hold the invariant $I$ fixed:

$$I(T(X)) = I(X), \quad X \in \mathfrak{A}.$$  

Key words: Matrices; invariants; determinant; generalized matrix function; rank.

1. Introduction

Let $K$ be a field and let $M_n(K)$ denote the vector space of all $n \times n$ matrices over $K$. Over the last 80 years, a great deal of effort has been devoted to the following question. Suppose $I(X)$ is an invariant defined on a subset $\mathfrak{A}$ of $M_n(K)$. Describe the set $\mathcal{L}(I, \mathfrak{A})$ of all linear transformations $T: \mathfrak{A} \rightarrow \mathfrak{A}$ that hold the invariant $I$ fixed:

$$I(T(X)) = I(X), \quad X \in \mathfrak{A}. \quad (1)$$

Even in this generality, it is clear that $\mathcal{L}(I, \mathfrak{A})$ is a multiplicative semigroup with an identity. The invariant $I$ can be a scalar valued function, e.g., $I(X) = \det(X)$; or for that matter it can describe a property, e.g., $\mathfrak{A}$ can equal $M_n(C)$ and $I(X)$ can mean that $X$ is unitary, so that we are simply asking for the structure of all linear transformations $T$ that map the unitary group into itself.

Much of a beginning course in linear algebra is devoted to the study of one aspect of this question for certain choices of $I$; for example, if $I(X) = \rho(X)$, the rank of $X$, then it is well known that the three standard linear operations on the rows and columns of a matrix leave $\rho$ fixed and this fact permits us to compute $\rho(X)$ by reducing $X$ to some normal form. The similarity theory is another example of this problem. In this case take $I(X)$ to be the set of all elementary divisors of the characteristic matrix of $X$, and then the linear operators $T$ that we wish to study are precisely those for which $I(X) = I(T(X))$.

In the survey paper [18, 1962] some of the aspects of this general problem are discussed. But since the time that paper was written there have been a number of developments. The purpose of this paper is to describe some of these.

2. Survey of Results

The scalar invariants are functions $I$ for which $I(X)$ is either an element of $K$ (we will assume that $\text{char } K = 0$, so that integer-valued functions are included) or a $p$-tuple of elements of $K$. Prob-
ably the first three problems of this kind were considered by Frobenius [11, 1897]:

(i) \( \mathcal{A} = M_n(C) \), \( I(X) = \det(X) \);

(ii) \( \mathcal{A} \) is the space of real-symmetric (or odd order skew-symmetric) matrices and \( I(X) = \det(X) \);

(iii) \( \mathcal{A} = \{ X \mid \text{tr}(X) = 0 \} \subset M_n(C) \) and \( I(X) = \det(X) \).

Frobenius proved what one might expect, namely that \( \mathcal{L}(\det, M_n(C)) \) consists of linear transformations of the form

\[
T(X) = UXV, \quad X \in M_n(C),
\]

or

\[
T(X) = UX^TV, \quad X \in M_n(C).
\]

In problem (i) \( \det(UV) = 1 \); in problem (ii) \( U = \xi A \), \( V = A^T \) where \( \xi \) is an appropriately chosen constant and \( \det A = 1 \); in (iii) \( U = \xi A \), \( V = A^{-1} \). Schur [34, 1925] extended and improved the result (i) as follows: Take \( \delta(X) \) to be the \( (\xi)^{2}\)-tuple of \( k^{th} \) order subdeterminants of \( X \) in some order where \( k \geq 3 \) is fixed; then Schur proved that if \( \delta(T(X)) = S(\delta(X)) \) for \( S \) a fixed nonsingular matrix, then \( T \) has one of the two forms (2) or (3) (without the restriction \( \det(UV) = 1 \)). Dieudonné [8, 1949] showed that if \( T \) is a semi-linear transformation of \( M_n(K) \) onto itself which holds the cone \( \det(X) = 0 \) invariant, then \( T \) is of the form

\[
T(X) = U(\sigma(x_{ij})) V, \quad X \in M_n(K),
\]

or

\[
T(X) = U(\sigma(x_{ij}))^T V, \quad X \in M_n(K),
\]

where \( \sigma \) is an automorphism of the field. In the paper [9, 1957] Dynkin states that the Frobenius theorems can be obtained using some results on the structure of maximal subgroups of the classical groups.

In an old result, Pólya [32] restricted \( T \) to be a linear transformation which affixes in a prescribed way + and - signs to the elements of \( X \), and asked whether such a \( T \) exists which satisfies \( \operatorname{per}(T(X)) = \det(X) \) for \( n > 2 \). Pólya answered the question negatively and many years later in [23, 1961] the question was answered negatively for arbitrary linear transformations \( T \). Along the lines of the Frobenius and Schur results, the structure of \( \mathcal{L}(E_r, M_n(C)) \) is determined where \( E_r(X) \) is the \( r \)th elementary symmetric function of the eigenvalues of \( X \), i.e., the sum of all \( r \)-square principle subdeterminants of \( X \). It was proved in [28, 1959] that for \( 4 \leq r < n \) any \( T \in \mathcal{L}(E_r, M_n(C)) \) is of the form

\[
T(X) = UXV, \quad X \in M_n(C),
\]

or

\[
T(X) = UX^TV, \quad X \in M_n(C),
\]

where \( UV = e^{i\varphi}I_n \) and \( r\varphi = 0 \) (2\pi). Just recently Beasley [1, 1970] completed the argument by showing that for \( r = 3 \), precisely the same result holds. E. P. Botta, in several papers considers the choice \( I(X) = d(X) \) where \( d \) is a generalized matrix function in the sense of Schur, i.e.,

\[
d(X) = \sum_{\sigma \in S_n} \lambda(\sigma) \prod_{i=1}^{n} x_{i, \sigma(i)},
\]

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where $\lambda$ is a nonzero function defined on a subgroup $H$ of $S_n$. In [3, 1967] Botta determined the structure of $\mathscr{L}(d, M_n(K))$ when $H$ is a transitive cyclic subgroup of $S_n$. In [4, 1968] $H$ is taken to be a doubly transitive or regular proper subgroup of $S_n$ and $\lambda$ is a character of $H$ of degree 1. In [5, 1967] Botta reprov'es an earlier result of Marcus and May [22, 1962] showing that $\mathscr{L}$ (per, $M_n(K)$) consists of precisely those $T$ of the form

$$T(X) = DPXQL, \quad X \in M_n(K),$$

or

$$T(X) = DPX^TQL, \quad X \in M_n(K),$$

where $P$ and $Q$ are permutation matrices, $D$ and $L$ are diagonal matrices and per $(DL) = 1$.

Many of the results concerning the structure of $\mathscr{L}(I, \mathfrak{A})$ can be reduced to the problem of determining linear maps on $M_n(K)$ which map the set of rank 1 matrices into itself. W. L. Chow [6, 1949], L. K. Hua [12, 1951] and Jacobson and Rickart [13, 1950] considered $1 - 1$ onto maps $T$ of $M_n(K)$ which have the property that both $T$ and $T^{-1}$ preserve coherence. In the present context, this amounts to assuming that $T$ and $T^{-1}$ both have the property that whenever $X$ and $Y$ differ by a matrix of rank 1, then $T(X)$ and $T(Y)$ differ by a matrix of rank 1. For linear maps this means $\rho(T(X)) = \rho(X)$ for all $X$. In [26, 1959] Marcus and Moyls proved that if $T: M_{m \times n}(K) \to M_{m \times n}(K)$ ($M_{m \times n}(K)$ is the space of all $m \times n$ matrices over $K$) is linear and $\rho(T(X)) = 1$ whenever $\rho(X) = 1$, then $T$ has the form

$$T(X) = UXV, \quad X \in M_{m \times n}(C),$$

or

$$T(X) = UX^TV, \quad X \in M_{m \times n}(C),$$

where $U$ and $V$ are fixed nonsingular matrices in $M_m(K)$ and $M_n(K)$ respectively. This result is fairly easy to apply because it does not require the a priori existence of $T^{-1}$. R. Westwick in [36, 1967] extended the result in [26, 1959] to linear maps on the space of $n$-contravariant tensors which hold the nonzero decomposable elements set-wise fixed. In another paper [35, 1964] Westwick, using techniques in [6, 1949] determined the structure of linear maps on the space $\Lambda^mV$ of skew-symmetric tensors into itself which hold the set of nonzero decomposable elements set-wise fixed. In a thesis at the University of British Columbia [7, 1967] L. Cummings proved that if $T$ maps the symmetric power $V^{(m)}$ into itself and holds the set of non-zero decomposable elements set-wise fixed then $T$ is induced by a linear map of $V$. Cummings’ result requires that the underlying field be algebraically closed of characteristic either 0 or exceeding $m$. In another thesis [14, 1971] M. H. Lim reconsiders this problem and relaxes the conditions on the field. Beasley [2, 1970] considered the problem of determining all linear transformations $T: M_n(K) \to M_n(K)$, $K$ algebraically closed, which hold the set of rank $k$ matrices set-wise fixed. Beasley required additional hypotheses on $T$ in order to prove that $T$ has the form (2) or (3). Djoković [10, 1969] proved that if $T$ maps the set of rank $k$ matrices into itself and is nonsingular, then in fact $T$ maps the set of rank 1 matrices into itself and the result in [26, 1959] applies. Much earlier [30, 1941] Morita proved that if $T$ maps the set of rank 1 matrices into itself and maps the set of rank 2 matrices into the set of matrices of rank at least 2, then $T$ has the form (2) or (3). He then used this to prove a result of Schur to the effect that if $I(X) = \alpha(X)$ is the Hilbert norm of $X$, i.e., the square root of the largest eigenvalue of $X^*X$, and $T \in \mathscr{L}(I, M_{m \times n}(C))$, then $T$ has the form (2) or (3) in which $U$ and $V$ are unitary. In a later paper [31, 1944], Morita shows that if $\mathfrak{A}$ is the set of $n$-square complex skew-symmetric matrices and $I(X)$ is again the Hilbert norm of $X$, then for $n \neq 4$ and $T \in \mathscr{L}(I, \mathfrak{A})$,

$$T(X) = U^TXU, \quad X \in M_n(C),$$

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where $U$ is a fixed unitary matrix; or if $n = 4$, then $T(X)$ can also have the alternative form

$$T(X) = UX^*U, \quad X \in M_n(C),$$

(4)

where $X^*$ is the matrix obtained from $X$ by interchanging the $(1, 4)$ and the $(2, 3)$ entry. A result similar to this was obtained by Westwick in his thesis [35, 1964]. In [29, 1960] Marcus and Westwick proved a theorem somewhat along the lines of the Morita theorem as follows. Let $k$ be a fixed integer satisfying $4 \leq 2k \leq n$. Let $\mathcal{V}$ be the set of skew-symmetric matrices over the field $\mathbb{R}$ of real numbers and let $T \in \mathcal{L}(E_{2k}, \mathcal{V})$. If $n \geq 5$, then there exists a real matrix $P$ such that

$$T(X) = \alpha PXPT, \quad X \in \mathcal{V},$$

(5)

where $\alpha PP^T = I_n$ is $2k < n$ and $\alpha PP^T$ is unimodular if $2k = n$. If $2k = n = 4$, then either $T$ has the form (5) or

$$T(X) = OP \begin{bmatrix} 0 & x_{34} & -x_{13} & -x_{14} \\ -x_{12} & 0 & x_{23} & -x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix} P^T,$$

where

$$X = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix},$$

and $\alpha PP^T$ is unimodular. Later on Marcus and Minc [24, 1962] proved that if $T \in \mathcal{L}(E'_r, M_{m,n}(C))$ where $1 < r \leq n$ and $E'_r(X)$ is the value of $E_r$ at the squares of the singular values of $X$, i.e., $E'_r(X)$ is just the value of $E_r$ at the eigenvalues of $X^*X$, then $T$ has the form (2) if $m \neq n$ and either (2) or (3) if $m = n$, where $U \in M_m(C)$ and $V \in M_n(C)$ are unitary.

In [17, 1959] it is proved that if $T$ is a linear map of $M_n(C)$ into itself such that $T(X)$ is unitary whenever $X$ is unitary, then $T$ is of the form (2) or (3) where $U$ and $V$ are unitary. B. Russo [33, 1969] recently used this result to prove the following interesting theorem. If $I(X)$ is the sum of the singular values of $X$ and if $T$ maps the identity matrix into itself, then $T \in \mathcal{L}(I, M_n(C))$ has the form (2) or (3), where $U$ and $V$ are unitary. Marcus and Gordon [20, 1970] recently proved the following result. Let $f(t) = f(t_1, \ldots, t_n)$ be a continuous, real-valued function defined for all $t_j \geq 0$, $1 \leq j \leq n$, and for $X \in M_{m,n}(C)$, let

$$I(X) = f(\alpha_1(X), \alpha_2(X), \ldots, \alpha_n(X))$$

where $\alpha_1(X) \geq \alpha_2(X) \geq \ldots \geq \alpha_n(X)$ are the singular values of $X$. If $f(t_1, \ldots, t_n)$ is concave, symmetric, strictly increasing in each $t_j$, and $f(0) = 0$, then $T \in \mathcal{L}(I, M_{m,n}(C))$ has the form (2) if $m \neq n$ and either (2) or (3) if $m = n$ where $U \in M_m(C)$ and $V \in M_n(C)$ are unitary. By specializing $f$ to

$$f(t) = \sum_{j=1}^{n} t_j^{\sigma}$$

where $0 < \sigma \leq 1$, the above theorem reduces to the following result. If $T: M_{m,n}(C) \to M_{m,n}(C)$ satisfies

$$\sum_{j=1}^{n} \alpha_j(T(X))^{\sigma} = \sum_{j=1}^{n} \alpha_j(X)^{\sigma}$$

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for all \( X \in M_{m,n}(C) \), then (2) or (3) holds with unitary \( U \) and \( V \). In the paper [27, 1970] the following result is proven using representation theory techniques. Let \( f(t) = f(t_1, \ldots, t_n) \) satisfy the conditions

(i) \( f(t) = 0 \) if and only if \( t = 0 \);
(ii) \( f \) is positively homogeneous of degree \( p \neq 0 \); i.e., \( f(ct) = c^p f(t) \), all \( c > 0, \ t \geq 0 \) (i.e., \( t_j \geq 0, j = 1, \ldots, n \)).

If \( I(X) = f(\alpha_1(X), \ldots, \alpha_n(X)) \) as before, then \( \mathcal{L}(I, M_{m,n}(C)) \) is a subgroup of the group of \( mn \times mn \) unitary matrices \( U(mn, C) \) where we associate each \( T \in \mathcal{L}(I, M_{m,n}(C)) \) with its matrix representation with respect to the lexicographically ordered basis \( \{ E_{st} = (\delta_{is}\delta_{jt}) \}, \ i, j = 1, \ldots, n \} \).

3. Current Work and Some Questions

M. J. S. Lim, in work closely related to that of Marcus and Westwick [29, 1960], has recently published [15, 1970] the following result. Let \( T \) map the space of skew-symmetric matrices over an algebraically closed field \( K \) into itself. Assume that \( T \) maps the set of rank 4 matrices into itself. Then for \( n = 4 \), \( T \) is of the form

\[
T(X) = \alpha PXP^T
\]

or

\[
T(X) = \alpha PX^tP^T.
\]

In case \( n = 4 \), \( T \) is one of the above forms, or else

\[
T(X) = \alpha PX^tP^T
\]

where \( X^t \) is defined in (4).

Just recently Marcus and Holmes [21, 1971] have proved the following results. Let \( X \in M_n(C) \). For any subgroup \( H \) of the symmetric group \( S_m \) of degree \( m \) and character \( \chi \) of degree 1 on \( H \), let \( K(X) : P \to P \) be the induced transformation [19, 1967] on the symmetry class of tensors \( (P, \nu) \) associated with \( H \) and \( \chi \). Define \( I(X) = tr K(X), X \in M_n(C) \).

(i) Let \( m \leq n \) or \( \chi = 1 \). \( \mathcal{L}(I, M_n(C)) \) is a group if and only if \( H \neq \{ e \} \).
(ii) Let \( H = S_m, \chi = 1 \) and \( \mathfrak{g} \subset M_n(C) \) an algebra with the property that \( \mathfrak{g}^* = \{ X^* : X \in \mathfrak{g} \} = \mathfrak{g} \), when \( X^* \) denotes the conjugate transpose of \( X \). Then \( \mathcal{L}(I, \mathfrak{g}) \) is a group.
(iii) In (i) take \( H = S_m, m \geq 3 \) and \( \chi = 1 \). If \( \mathcal{L}(I, M_n(C)) \) denotes the subgroup of \( \mathcal{L}(I, M_n(C)) \) of those \( T : M_n(C) \to M_n(C) \) satisfying \( T(I_n) = \xi I_n \), then for any \( T \in \mathcal{L}(I, M_n(C)) \) there exists a fixed nonsingular matrix \( P \in M_n(C) \) such that

\[
T(X) = \xi P^{-1}XP, \quad X \in M_n(C),
\]

or

\[
T(X) = \xi P^{-1}X^tP, \quad X \in M_n(C).
\]

In this case \( tr K(X) \) is the completely symmetric function of the eigenvalues of \( X \), denoted here by \( h_m(X) \). Thus this result states that if \( T(I_n) = \xi I_n \) and \( h_m(T(X)) = h_m(X) \) for all \( X \in M_n(C) \) then \( T \) has the form (6) or (7).

(iv) In (i) take \( A_m \subset S_m \) to be the alternating group, \( m \geq 3 \) and \( \chi = 1 \). Then the group \( \mathcal{L}(I, M_n(C)) \) consists precisely of those linear transformations \( T \) of the form (6) or (7).

There are a number of questions which remain unanswered. For example, a more direct proof of the result in [27, 1970] might be based on the following.
**Conjecture 1:** Let $T$ be an $mn$-square complex matrix, and assume that for arbitrary unitary matrices $U \in M_m(\mathbb{C})$, $V \in M_n(\mathbb{C})$ the matrix $(U \otimes V)T$ has eigenvalues of modulus 1. (The matrix $U \otimes V$ is the usual Kronecker product of $U$ and $V$.) Then $T$ is unitary.

**Conjecture 2:** If $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear map and if $h_m(T(X)) = h_m(X)$, $X \in M_n(\mathbb{C})$ (recall that $h_m(X)$ is the completely symmetric function of the eigenvalues of $X$), then in fact $T(I_n) = \xi I_n$ and hence from [21, 1971] $T$ has the form (6) or (7).

**Conjecture 3:** Let $P_m(X)$ denote the $m$th induced power matrix of $X$ [25] and suppose that $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies $P_m(T(X)) = S(P_m(X))$, $X \in M_n(\mathbb{C})$ where $S : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a fixed non-singular linear map, $N = \left(\frac{m + m - 1}{m}\right)$. Then $T$ has the form (2) or (3).

**Conjecture 4:** Suppose $K(X)$ is an invariant matrix [16, Chapter X] defined by means of a Young tableau. If $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\tr K(T(X)) = \tr K(X)$ for all $X \in M_n(\mathbb{C})$, then $T$ must have the form (2) or (3) with some appropriate conditions on the $U$ and $V$. Of course, the result in [28, 1959] and [21, 1971] are special cases of this.

As a possible extension of Schur’s theorem [30, 1941] consider

**Conjecture 5:** Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $h_m(T(X^*T(X)) = h_m(X^*X)$ (see Conjecture 2 in which $h_m$ is defined), then $T$ has the form (2) or (3).

As a variant of the result in [17, 1959], let $G$ be any of the following classical groups: the real orthogonal group, the rotation group, the symplectic group.

**Conjecture 6:** Let $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ map $G$ into itself. Then $T$ must have the form (2) or (3) in which $U$ and $V$ belong to $G$. (In the case of the rotation group, $U$ and $V$ could be simply real orthogonal with $\det (UV) = 1$.)

**Conjecture 7:** Suppose $T$ is a mapping of the space of 2-contravariant tensors into itself; and suppose moreover that for each decomposable element $x \otimes y$ we have $\|T(x \otimes y)\| = \|x \otimes y\|$ (Euclidean norm). Then $T$ is unitary. This can be restated in terms of linear maps $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$. Thus suppose for $\rho(X) = 1$, $\|T(X)\| = \|X\|$ where $\|X\|$ is just the Euclidean norm. Show that $T$ is unitary.
4. References


(Paper 75B3 & 4–350)