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The Powers of a Connected Graph are Highly Hamiltonian*

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Let G be a connected graph on p points. It is proved that if any k points are deleted from the graph G^{m+2}, $1 \leqslant m \leqslant p-3$ and $0 \leqslant k \leqslant m$, then the resulting graph is hamiltonian.

Key words: graph, hamiltonian, power of a graph.

Let G be a graph (finite, undirected, without loops or multiple lines). A *walk* of G is a finite alternating sequence of points and lines of G, beginning and ending with points, and where each line is incident with the two points immediately preceding and following it. A walk in which all points are distinct is called a *path*; the *length* of a path is the number of lines in it. A graph G is said to be *connected* if between any two points of G there is a path. The *distance* between two points u and v in a connected graph G is the length of a shortest path between u and v if $u \neq v$, and zero if $u = v$.

A walk with at least three distinct points in which the first and the last points are the same but all the other points are distinct is called a *cycle*. A cycle containing all the points of G is a *hamiltonian cycle* of G, and then G itself is said to be a *hamiltonian graph*. A graph G on p points is *m-hamiltonian* if the removal of any k points from G, $0 \leqslant k \leqslant m \leqslant p-3$, yields a hamiltonian graph (see [2]).[1]

Let G be a connected graph and $n \geqslant 1$. The graph G^n, called the nth *power of* G, has as its points those of G, and two distinct points u and v are adjacent in G^n if the distance between u and v in G is at most n.

For a connected graph G on $p \geqslant 3$ points, it is well known that G^3 is hamiltonian; in fact (see [1]) if x is any line of G, there is a hamiltonian cycle of G^3 which contains x. The main result of [1] was: the cube of every connected graph possessing at least 4 points is 1-hamiltonian. It is this result which we generalize as follows.

THEOREM: *If* G *is a connected graph on* p *points, then* G^{m+2} *is* m-*hamiltonian for* $1 \leqslant m \leqslant p-3$.

The proof is by construction.

To start with we make some observations. As G^3 is hamiltonian, and $G^3 \subseteq G^4 \subseteq \ldots \subseteq G^{m+2}$, it suffices to prove that the deletion of any $n \geqslant 1$ points from G^{n+2} yields a hamiltonian graph. Moreover, since every connected graph contains a spanning tree, without loss of generality we may assume G to be a tree. The case $m=1$ is dealt with in [1]. So we take $m \geqslant 2$ for the rest of the discussion.

Let T be a tree on p points and let m be such that $2 \leqslant m \leqslant p-3$. Let $V(m) = \{u_1, u_2, \ldots, u_m\}$ be a set of m distinct points of T. Denote the unique path in T joining the points u_i and u_j of $V(m)$ by $P(u_i, u_j)$. We define the tree $\tilde{T}$ to be the subtree of T spanned by the points of all the paths $P(u_i, u_j)$, $1 \leqslant i, j \leqslant m$.

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[1] Figures in brackets indicate the literature references at the end of this paper.

For the sake of clarity we assign colors to the points of T: the points of $V(m)$ are colored *red*, those of $\tilde{T}-V(m)$ *green*, and the remaining points of T *blue*. Obviously every endpoint of $\tilde{T}$ is red. Designate one of the endpoints of the nontrivial tree $\tilde{T}$ as the *initial* point. The remaining endpoints of $\tilde{T}$ will be referred to as *reversing* points. For a point u of $\tilde{T}$ we denote by $d(u)$ and $\tilde{d}(u)$, the degree of u in T and $\tilde{T}$ respectively. If $\tilde{d}(u) \geqslant 3$, then u will be called a *crucial* point, and this may be red or green.

The $n(\geqslant m)$ points of $\tilde{T}$ are labeled $1, 2, \ldots, n$ in the following manner (see fig. 1). The initial point of $\tilde{T}$ is labeled 1. The unique point of $\tilde{T}$ adjacent to the initial point is labeled 2. Suppose that the labels $1, 2, \ldots, i$ have been assigned, $1 < i \leqslant n$. If $\tilde{d}(i) \geqslant 3$, i.e., if i is a crucial point, then a point of $\tilde{T}$ adjacent to i and not already labeled is designated $i+1$. If $\tilde{d}(i)=2$, then the unique nonlabeled point of T adjacent to i is labeled $i+1$. If $\tilde{d}(i)=1$, then i is a reversing point. If there are no crucial points on the path in $\tilde{T}$ between 1 and i, then $\tilde{T}$ itself is a path, and it follows that $i=n$. If not, let $S(i)=(j_1, j_2, \ldots, j_{r_i})$ be the sequence of all the crucial points between 1 and i, and such that $1 < j_1 < j_2 < \ldots < j_{r_i} < i$. If there is a nonlabeled point of $\tilde{T}$ adjacent to j_{r_i}, then we label it $i+1$. If not, we consider the nonlabeled points of $\tilde{T}$ adjacent to j_{r_i-1}, j_{r_i-2}, and so on till j_1. If even j_1 has all the points adjacent to it in $\tilde{T}$ already labeled, then clearly $i=n$.

As already noted, if $\tilde{T}$ has at least one crucial point, then with every reversing point i of $\tilde{T}$ we can associate a unique *crucial sequence* $S(i)$ as defined above. If $\tilde{T}$ is a path, write $S(n)=\emptyset$. Moreover, if j is a crucial point, then there are at least $\tilde{d}(j)-1$ reversing points whose associated crucial sequences contain j. (See fig. 1.).

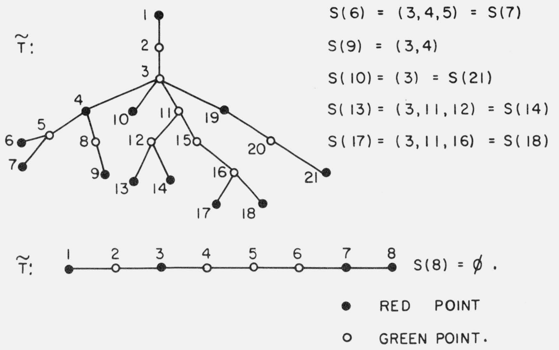

FIGURE 1. *Examples of trees* T̃; *their labeling and crucial sequences.*

Every point i of $\tilde{T}$ is adjacent to $d_i = d(i)-\tilde{d}(i) \geqslant 0$ blue points of T. Let the forest in $T-i$ to which these d_i points belong be denoted F_i. For the sake of completeness, if $d_i=0$, we will reflect this by writing $F_i=\emptyset$. If $d_i \geqslant 1$, let the d_i components of F_i be the trees T_t on p_t points, $1 \leqslant t \leqslant d_i$. (See fig. 2.) Let x_t be the point in T_t which is adjacent to i, and x_t' a point in T_t which is adjacent to x_t (if $p_t=1$, we assign a second label x_t' to x_t). Since $m \geqslant 2$, the set of points $\{x_t, x_t' | 1 \leqslant t \leqslant d_i\}$ induce a complete graph in T^{m+2}.

Consider the graph $T_t^3 \subseteq T^{m+2}$. We have

(i) if $p_t \geqslant 3$, then by earlier remarks there exists a hamiltonian cycle in T_t^3 containing the line $x_t x_t'$, and by deleting this line from this cycle there results a spanning path P_t in T_t^3 such that x_t and x_t' are the endpoints of P_t,

(ii) if $p_t=2$, T_t is itself the path P_t, and

(iii) for $p_t=1$, $x_t=T_t=P_t$.

It is now possible to construct a special path $P(i)$ in T^{m+2} which spans F_i. Define

$$P(i)=\begin{cases} x_1P_1x_1' \quad \text{or} \quad x_1'P_1x_1, & \text{if} \quad d_i=1 \\ x_1P_1x_1'x_2P_2x_2'x_3 \; \ldots\ldots \; x_{d_i-1}P_{d_i-1}x'_{d_i-1}x'_{d_i}P_{d_i}x_{d_i}, & \text{if} \quad d_i \geqslant 2, \end{cases}$$

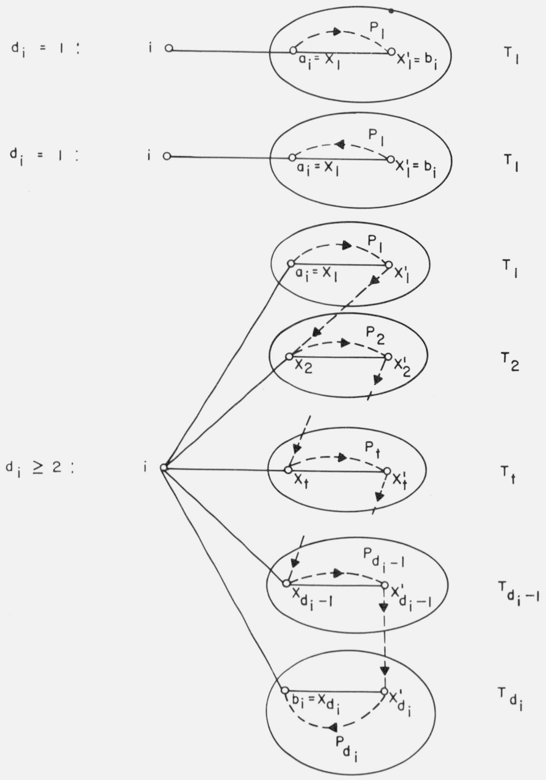

FIGURE 2. *The forest* F_i *and its spanning path.*

where $x_kP_kx_k'$ refers to the path P_k starting at x_k and ending at x_k' (similarly for $x_k'P_kx_k$). If $F_i \neq \emptyset$, we are only concerned with the starting and ending points of the spanning path $P(i)$ of F_i. Towards this end we set

$$a_i = x_1$$

and

$$b_i = \begin{cases} x_1' & \text{if} \quad d_i = 1 \\ x_{d_i} & \text{if} \quad d_i \geqslant 2. \end{cases}$$

Then $P(i;\, a_i,\, b_i)$ will denote the path $P(i)$ starting at a_i and ending at b_i; whereas $P(i;\, b_i,\, a_i)$ will be the path $P(i)$ beginning at b_i and ending at a_i.

We now construct a hamiltonian cycle C in $T^{m+2} - V(m)$. First, an outline will be suggested. Starting at the initial point 1, we move around the points of the tree $\tilde{T}$ in the order of its labeling. At each point i with $F_i \neq \emptyset$, the spanning path $P(i;\, a_i,\, b_i)$ or $P(i;\, b_i,\, a_i)$ of F_i is included in C in a specific order. For green points of $\tilde{T}$ having $F_i = \emptyset$, a special procedure is adopted. Picking on C the first such green point encountered, the remaining are picked alternately while going towards a reversing point. Those missed, as well as the green points with $F_i \neq \emptyset$, are picked up while returning (through crucial points) from a reversing point to the next point of $\tilde{T}$ in its labeling. The residual green points are picked while going from the last reversing point n to the starting point of C. Clearly, C contains all the nonred points of T, and is in fact a cycle. Also, the adjacencies in T^{m+2} are used to proceed from the considerations at the ith stage to the next.

First we settle the only extreme case. Suppose that $\tilde{T}_0$ is a path on $n = m+1$ points, $d_1 = d_n = 1$, and $F_i = \emptyset$ for $2 \leqslant i \leqslant n-1$ (see fig. 3). Let g denote the only green point of $\tilde{T}_0$. Then

$$C = \underline{P(1;\, a_1,\, b_1)gP(n;\, b_n,\, a_n)a_1}.$$

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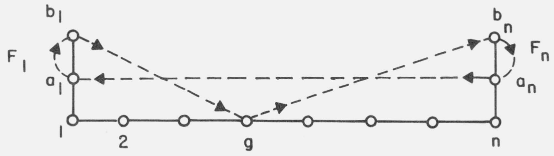

FIGURE 3. $\tilde{T}_0$ *with* $n=m+1$, $d_1=d_n=1$.

Now suppose that $\tilde{T} \neq \tilde{T}_0$. We construct the hamiltonian cycle C as follows, keeping in mind that the first point that is picked on C will be considered as its starting point.

Step 1: Consider $1 \in \tilde{T}$. If $F_1 \neq \emptyset$, pick $P(1;\, a_1,\, b_1)$ and proceed to the next step. If $F_1 = \emptyset$, we go to step 2.

Step 2: $F_2 \neq \emptyset$: Pick $P(2;\, a_2, b_2)$ and proceed to step 3. For $n=2$, either $\underline{P(1;\, a_1,\, b_1)P(2;\, a_2,\, b_2)a_1}$ or $\underline{P(2;\, a_2,\, b_2)a_2}$ is the cycle C.

$F_2 = \emptyset$ and the point 2 is red: Proceed to step 3 directly. If $n=2$, then $\underline{P(1;\, a_1,\, b_1)a_1}$ is C.

$F_2 = \emptyset$ and the point 2 is green: Pick 2 on C and proceed to the next step.

Continuing this process, suppose we have considered the points 1, 2, . . ., $i-1$, i.e., *step* $i-1$ has been taken.

Step i: Suppose $\tilde{d}(i) \geqslant 2$.

$F_i \neq \emptyset$: Pick $P(i;\, a_i,\, b_i)$ and proceed to the next step.

$F_i = \emptyset$ and the point i is red: Proceed to step $i+1$.

$F_i = \emptyset$ and the point i is green: Pick i on C if an immediately preceding green point j with $F_j = \emptyset$ has not been picked and then move to step $i+1$, otherwise go to step $i+1$ directly.

Suppose $\tilde{d}(i) = 1$.

Here i is a reversing point.

Let $i \neq n$. If $F_i \neq \emptyset$, pick $P(i;\, a_i,\, b_i)$ (if $F_i = \emptyset$, this is absent) and pick up in reverse succession all the unpicked green points on the path in $\tilde{T}$ between the points i and $i+1$.

Let $i=n$. If $F_n \neq \emptyset$, pick $P(n;\, a_n,\, b_n)$ (if $F_n = \emptyset$, this is absent) and pick up on C, all the remaining green points in reverse succession towards 1, and reach the starting point of C.

This completes the construction.

The result proved above is the best possible in the sense that the graph G^{m+2} is not necessarily $(m+1)$-hamiltonian. For example, let P be a path on $m+4$ points, $m \geqslant 1$. Let $V(m+1)$ be any set of $m+1$ points of P, excluding the end points. Then the graph $P^{m+2} - V(m+1)$ is a path on 3 points and clearly nonhamiltonian. On the other hand, if G is a connected graph on $p \geqslant 3$ points, then G^n is the complete graph K_p for all $n \geqslant p-1$, and hence it is always $(p-3)$-hamiltonian (see [2]).

The theorem does not hold for $m=0$, i.e., the square of a connected graph need not be hamiltonian. But if *connected* is replaced by *block*, the proposition reads—*the square of a block on 3 or more points is hamiltonian.* M. D. Plummer has conjectured that this is true.

Added in proof. H. Fleishner has confirmed Plummer's conjecture.

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