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A Theorem on Matrix Commutators

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Let $P=I_p\dot{+}(-I_q)$, the direct sum of the $p\times p$ identity matrix and the negative of the $q\times q$ identity matrix. The following theorem is proved.

THEOREM: *If the matrices* A *and* B *are* P*-orthogonal (orthogonal with respect to* P*) and* P*-skew-symmetric (skew-symmetric with respect to* P*) then* [A, B]=AB−BA *may be expressed as the difference,* C−D, *of matrices which are* P*-orthogonal,* P*-skew-symmetric.*

Explicit expressions for the matrices C and D are given.

Key words: Commutator; matrix; orthogonal; skew-symmetric.

1. Introduction

Let $P=I_p\dot{+}(-I_q)$, the direct sum of the $p\times p$ identity matrix and the negative of the $q\times q$ identity matrix. Katz and Olkin [1][2] define a matrix A to be orthogonal with respect to P (P-orthogonal) if only if

$$APA'=P \tag{1}$$

where A' is the transpose of the matrix A. Furthermore, they define a matrix B to be skew-symmetric with respect to P (P-skew-symmetric) if and only if the matrix BP is skew-symmetric in the ordinary sense.

The results of this paper are concerned with matrices which are both P-orthogonal and P-skew-symmetric of order $n=p+q$. J. M. Smith [4] proves that such matrices exist if and only if both n and p are even. Consider the following:

THEOREM 1: *If the* $n\times n$ *matrices* A *and* B *are both* P*-orthogonal and* P*-skew-symmetric then* $[A, B]\equiv AB-BA$ *is a scaler multiple of a* P*-orthogonal,* P*-skew-symmetric matrix.*

The theorem has been proved in the cases $n=p=4$ and $n=p=2$ by Pearl [2] and in the case $n=4$, $p=2$ by J. M. Smith [3]. However, the above theorem is not true for all values of n and p. For example, let $n=p=6$ and let:

$$A=\begin{pmatrix}0&0&0&0&0&-1\\0&0&-1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&-1&0&0\\1&0&0&0&0&0\end{pmatrix},\quad B=\begin{pmatrix}0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\\0&0&0&0&-1&0\end{pmatrix}.$$

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[2] Figures in brackets indicate the literature references at the end of this paper.

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Thus Theorem 1 is not true without restrictions being placed on n and p.

In this paper, the following general theorem is proved:

THEOREM 2: *If* A *and* B *are both* P-*orthogonal*, P-*skew-symmetric matrices then*

$$[A, B] \equiv AB - BA = C - D$$

where C *and* D *are both* P-*orthogonal*, P-*skew-symmetric matrices.*

Theorem 2 is true for all values of n and p. Furthermore, explicit solutions are given for C and D in terms of A and B.

2. Proof of Theorem 2

Before proceeding, it is convenient to note the following:

LEMMA: *If the matrix* A *is* P-*skew-symmetric, then* $A' = -PAP$.

PROOF: Since A is P-skew-symmetric, AP is skew-symmetric and thus, $(AP)' = -AP$ or $P'A' = -AP$. But $P' = P$ and $P^2 = I$. Hence, $A' = -PAP$.

In order to prove Theorem 2, consider writing $[A, B]$ as $[A, B] = (AB+X) - (BA+X)$. The investigation of requisite properties of the matrix X in order that $AB+X$ and $BA+X$ be both P-orthogonal and P-skew-symmetric leads to the following result.

THEOREM 3: *Let* A *and* B *be* P-*orthogonal*, P-*skew-symmetric matrices. Then* AB + X *and* BA + X *are* P-*orthogonal*, P-*skew-symmetric matrices if and only if all of the following hold:*

(a)
$$XPX' = -(ABPX' + XPB'A')$$
$$= -(BAPX' + XPA'B')$$

(b)
$$X + PX'P = -AB - BA$$

(c)
$$X^2 = -P(X^2)'P.$$

PROOF: Assume $AB+X$ and $BA+X$ to be P-orthogonal, P-skew-symmetric matrices. Since $AB+X$ is P-orthogonal,

$$(AB+X)\, P\, (AB+X)' = P$$

or

$$ABPB'A' + ABPX' + XPB'A' + XPX' = P$$

and by (1), $ABPB'A' = P$.

Hence,

$$ABPX' + XPB'A' + XPX' = 0. \tag{2}$$

Similarly since $BA+X$ is P-orthogonal

$$BAPX' + XPA'B' + XPX' = 0. \tag{3}$$

Equations (2) and (3) form condition (a). Furthermore, since $AB+X$ is P-skew-symmetric,

$$((AB+X)P)' = -(AB+X)P$$

or

$$P'B'A' + P'X' = -ABP - XP. \tag{4}$$

by the Lemma, $A' = -PAP$ and $B' = -PBP$ and since $P^2 = I$, (4) becomes

$$BAP + PX' = -ABP - XP.$$

Multiplying on the right by P,

$$BA+PX'P=-AB-X$$

which is condition (b). Similarly, the fact that $BA+X$ is P-skew-symmetric leads to condition (b). Finally, adding (2) and (3) one obtains

$$2XPX'+(AB+BA)\,PX'+XP\,(A'B'+B'A')=0$$

and applying condition (b) to the factors $AB+BA$ and $A'B'+B'A'$; gives condition (c).

Conversely, let a matrix X satisfy conditions (a), (b) and (c). By (c)

$$-X^2P-PX^{2\prime}=0$$

so that

$$2XPX'-XPX'-PX'^2-XPX'-X^2P=0$$

or

$$2XPX'+(-X-PX'P)\,PX'+XP\,(-X'-PXP)=0. \tag{5}$$

Applying condition (b) to (5),

$$2XPX'+(AB+BA)\,PX'+XP\,(A'B'+B'A')=0$$

or

$$(XPX'+ABPX'+XPB'A')+(XPX'+XPA'B'+BAPX')=0.$$

By condition (a) both of these terms are individually equal to zero. Thus it follows that $AB+X$ and $BA+X$ are P-orthogonal. Similarly, reversing the steps which led to condition (b), it follows that $AB+X$ and $BA+X$ are P-skew-symmetric.

It follows from Theorem 3 that if such a matrix X can be found the proof of Theorem 2 is completed. Letting $Y=XP$, then $Y'=PX'$ and multiplying (b) on the right by P

$$Y+Y'=-ABP-BAP.$$

Furthermore Y has a unique representation as the sum $S+Q$ of a symmetric matrix S and a skew-symmetric matrix Q. This leads to

Corollary 1: *The symmetric part of* Y *is uniquely given by*

$$\mathrm{S}=-\tfrac{1}{2}(\mathrm{AB}+\mathrm{BA})\,\mathrm{P}.$$

The skew-symmetric part of Y, Q, is not as easily found, however, certain conditions may be noted.

From (b) it follows that

$$X'=-PXP-PABP-PBAP$$

and substituting in both parts of (a) for X' leads to

$$XP(-PXP-PABP-PBAP)=ABXP+ABABP+ABBAP-XPB'A'$$

and

$$XP(-PXP-PABP-PBAP)=BAXP+BAABP+BABAP-XPA'B'.$$

Equating the expressions on the right and grouping appropriately leads to

$$[A,B]XP+XP[A',B']=BAABP-ABBAP+(BA)^2P-(AB)^2P.$$

Again, letting $Y=XP$ and noting that $A^2=-I$ and $B^2=-I$,

$$[A, B]Y-Y[A', B']=(BA)^2P-(AB)^2P.$$

Writing Y as $S+Q$, applying the fact that $A'B'=PABP$, and simplifying leads to

$$[A, B]Q+Q[A', B']=0.$$

Furthermore, by (c) $X^2=-P(X^2)'P$ and since $Y=XP$ then $YP=X$ or

$$(YP)^2=-P(YP)^{2\prime}P.$$

But $YP=SP+QP$ and hence

$$(SP)^2+(QP)^2+SPQP+QPSP=-(SP)^2-(QP)^2+QPSP+SPQP$$

or

$$(QP)^2=-(SP)^2.$$

Thus, by Corollary 1,

$$(QP)^2=-\tfrac{1}{4}(ABAB+BABA+2I).$$

This may be summarized as:

COROLLARY 2: *The skew-symmetric part* Q *of the matrix* Y *must satisfy both of the following conditions:*

(a) $$[\mathrm{A}, \mathrm{B}]\mathrm{Q}+\mathrm{Q}[\mathrm{A}', \mathrm{B}']=0$$

(b) $$(\mathrm{QP})^2=-\tfrac{1}{4}(\mathrm{ABAB}+\mathrm{BABA}+2\mathrm{I}).$$

While this does not uniquely define the matrix Q, it can be easily shown that one solution is

$$Q=\tfrac{1}{2}(ABABA+A)P.$$

Since $Y=S+Q$ and $X=YP$ it follows that the matrix

$$X=\tfrac{1}{2}(-AB-BA+ABABA+A)$$

satisfies all three conditions of Theorem 3.

Since $AB+X=\frac{1}{2}(AB-BA+ABABA+A)$ and $BA+X=\frac{1}{2}(-AB+BA+ABABA+A)$, Theorem 2 may be restated as a corollary of Theorem 3.

COROLLARY 3: Let A and B be P-orthogonal, P-skew-symmetric matrices. Then $[A,B]=C-D$ where C and D are P-orthogonal, P-skew-symmetric matrices. Furthermore,

$$C=\tfrac{1}{2}(AB-BA+ABABA+A),$$
$$D=\tfrac{1}{2}(-AB+BA+ABABA+A).$$

It should be noted that a second expression of the matrices C and D is obtained by interchanging A and B in the explicit expression given for C and D in the Corollary 3.

3. Conclusion

The main result of this paper, Theorem 2, may be stated as: Given any two P-orthogonal, P-skew-symmetric matrices A and B their commutator, $[A, B]$, may always be expressed as a linear combination (difference) of two other such matrices. Also, Corollary 3 gives an explicit representation of this linear combination.

An open question remains with regard to the matrix X defined in Theorem 3. Corollary 1 shows that the symmetric part of XP is unique. However, the skew-symmetric part of XP is not unique, and two expressions have been noted. Do other expressions exist and how may they be characterized?

4. References

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