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Contractifiable Semigroups

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Consider a continuous semigroup of operators on a metric space, indexed $\{T_t\}$ by the nonnegative real numbers. It is shown that if any one of the operators can be made into a contraction by some topology-preserving remetrization, then for each $\lambda \epsilon (0, 1)$ there is a metric under which each operator $T_t (t > 0)$ becomes a contraction with contraction constant λ^t. With the operators regarded as describing the evolution of an autonomous dynamical system, this metric can be used to define a Lyapanov function.

Key words: Contractions; functional analysis; operators; semigroups; stability theory.

1. Introduction

This paper is the fifth in a series [1–4],[1] dealing with the question of when continuous self-mappings, of a metrizable topological space X, can be made into contractions by choosing a suitable metric for X. A single map f with this property will be termed *contractifiable*. It was shown in [2] that such maps f are characterized by the properties one would expect in view of Banach's Contraction Principle, namely the existence in X of a fixed point ξ of f and an open neighborhood U of ξ such that [2]

$$f^n(x) \to \xi \qquad \text{(all } x \epsilon X), \tag{1.1}$$

$$f^n(U) \to \{\xi\}. \tag{1.2}$$

Now let $\mathscr{T}$ be a commutative *family* of self-mappings of X, with common fixed-point ξ. Assume the members of $\mathscr{T}$ are individually contractifiable, so that there are collections $\{d_f : f \epsilon \mathscr{T}\}$ and $\{\lambda_f : f \epsilon \mathscr{T}\}$ such that each $f \epsilon \mathscr{T}$ is a contraction mapping of metric space (X, d_f) with $\lambda_f \epsilon (0, 1)$ as contraction constant. If d_f can be chosen independent of f, we term $\mathscr{T}$ *simultaneously contractifiable*. In [3] it is shown that $\mathscr{T}$ will indeed be simultaneously contractifiable, if it is *finite*. The present paper provides the analogous result for the most important special case of infinite families $\mathscr{T}$.

The special case is given by $\mathscr{T} = \mathscr{S} - \{T_0\}$, where $\mathscr{S} = \{T_t : t \epsilon R^+\}$ is an *R^+-semigroup of operators* on metric space (X, d). Here R^+ is the topological semigroup of nonnegative real numbers under addition, d is a metric on X (the "original" metric, prior to remetrization), and each T_t is a continuous self-mapping of X. The terminology means that T_0 is the identity map of X, that the semigroup property $T_{t+s} = T_t T_s$ is satisfied, and that the continuity property

$$\sup \{d(T_t(x), T_s(x)) : x \epsilon X\} \to 0 \qquad (\text{as } s \to t) \tag{1.3}$$

holds for all $t \epsilon R^+$. Our main result is the following.

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[1] Figures in brackets indicate the literature references at the end of the paper.

[2] To avoid inessential technical complications we assume throughout that ξ actually lies in X, instead of being an "ideal point" corresponding to a class of equivalent Cauchy sequences.

THEOREM: *Let* $\mathscr{S}=\{T_t : t\epsilon R^+\}$ *be an* R^+*-semigroup of operators on* (X, d). *If any member* T_τ *of* $\mathscr{S}$ *is contractifiable, then for each* $\lambda\epsilon(0, 1)$ *there is a topologically equivalent metric* ρ_λ *such that each* $T_t(t > 0)$ *is a contraction on* (X, ρ_λ) with λ^t *as contraction constant.*

This result must be distinguished from that in [4], where it was shown that there existed metrics $\rho_\lambda^{(t)}$ such that each $T_t(t > 0)$ is a contraction on $(X, \rho_\lambda^{(t)})$ with λ as contraction constant. In particular, although the relationship of our topic with asymptotic stability was worked out in Theorem 2 of [4], it is only the above theorem which makes evident the association with a Lyapanov function for the semigroup $\mathscr{S}$ and the point $\xi\epsilon X$.

In this stability-theoretic context, X is interpreted as the space of possible "states" of an autonomous dynamical system, while $T_t(x)$ is interpreted as the system's state at time t if its initial state was x. The fixed-point property of ξ characterizes it as an "equilibrium state," whose stability is to be studied. In terms of the metric ρ_λ of the above Theorem, set

$$L(x) = \rho_\lambda(\xi, x).$$

Then L is continuous, and positive-definite with respect to ξ. Since

$$L(T_{s+t}(x)) \leqslant \lambda^s L(T_t(x)),$$

L is nonincreasing along every trajectory, and strictly decreasing unless and until the trajectory reaches ξ. These properties identify L as a Lyapanov function.

2. Two Lemmas

The analysis will employ the notation

$$S(x, r) = \{y\epsilon X : d(x, y) \leqslant r\}. \tag{2.1}$$

It is convenient to precede the main proof by the following two lemmas, the first of which is a slight generalization of the lemma in [4].

LEMMA 1: *Given* $x\epsilon X$, $t > 0$, *and* $\eta > 0$, *there exists* $\delta > 0$ *such that*

$$T_s(S(x, \delta)) \subseteq S(T_s(x), \eta) \qquad \textit{for all } s\epsilon[0, t].$$

PROOF: If not, then there exist sequences $\delta_n \to 0$, $s(n)\epsilon[0, t]$, and $x_n\epsilon S(x, \delta_n)$ such that

$$d(T_{s(n)}(x), T_{s(n)}(x_n)) > \eta. \tag{2.2}$$

By passing to a subsequence, we may assume $s(n) \to s\epsilon[0, t]$. Then

$$d(T_{s(n)}(x), T_{s(n)}(x_n)) \leqslant d(T_{s(n)}(x), T_s(x)) + d(T_s(x), T_s(x_n)) + d(T_s(x_n), T_{s(n)}(x_n)).$$

By the continuity condition (1.3) and the fact $s(n) \to s$, the first and third summands on the right tend to 0 as $n \to \infty$; since T_s is continuous and $\delta_n \to 0$ implies $x_n \to x$, the same is true of the second summand. Thus the last display leads to a contradiction of (2.2).

LEMMA 2: *If some* T_τ *is contractifiable, then there is an open neighborhood* U *of* ξ *such that*

$$T_t(U) \subseteq U \qquad \textit{for all } t \geqslant 0, \tag{2.3}$$

$$T_t(U) \to \{\xi\} \qquad \textit{as } t \to \infty. \tag{2.4}$$

PROOF: Since T_τ is contractifiable, there is an open neighborhood U_τ of ξ such that $T_\tau^n(U_\tau) \to \{\xi\}$ as $n \to \infty$. As shown in [2], it can further be assumed that $T_\tau(U_\tau) \subseteq U_\tau$. For each $s \epsilon [0, \tau]$, define the open set

$$U_s^* = \{x \epsilon X : T_s(x) \epsilon U_\tau\}$$

and put

$$U = \cap \{U_s^* : s \epsilon [0, \tau]\}.$$

To prove (2.3) for any given $t \geqslant 0$, choose any $s \epsilon [0, \tau]$ and write

$$t + s = n\tau + \sigma \qquad (\sigma \epsilon [0, \tau];\ \text{integral } n \geqslant 0).$$

Then since $U \subset U_\sigma^*$, we have

$$T_{t+s}(U) = T_\tau^n T_\sigma(U) \subseteq T_\tau^n(U_\tau) \subseteq U_\tau,$$

so that $T_t(U) \subseteq U_s^*$. Hence (2.3) holds.

To prove (2.4), consider any $\eta > 0$. By Lemma 1, there is a $\delta > 0$ such that

$$T_s(S(\xi, \delta)) \subseteq S(\xi, \eta) \qquad \text{for all } s \epsilon [0, \tau].$$

Choose N so large that

$$T_\tau^n(U_\tau) \subseteq S(\xi, \delta) \qquad \text{for all } n \geqslant N.$$

Then for $t \geqslant t_0(\eta) = N\tau$ we have

$$t = n\tau + \sigma \qquad (n \geqslant N;\ \sigma \epsilon [0, \tau)),$$

and so

$$T_t(U) \subseteq T_t(U_0^*) = T_t(U_\tau) = T_\sigma T_\tau^n(U_\tau)$$

$$\subseteq T_\sigma(S(\xi, \delta)) \subseteq S(\xi, \eta).$$

This establishes (2.4).

Clearly $\xi \epsilon U$; to prove U an open neighborhood of ξ, it remains to show that U is open. Consider any $x \epsilon U$. We assert the existence of a $\delta > 0$ such that

$$T_s(S(x, \delta)) \subseteq U_\tau \qquad \text{for all } s \epsilon [0, \tau]\,; \tag{2.5}$$

this implies $S(x, \delta) \subseteq U$, and thus that U is open. If no such δ existed, there would be sequences $\delta_n \to 0$, $s(n) \epsilon [0, \tau]$ and $x_n \epsilon S(x, \delta_n)$ such that $T_{s(n)}(x_n) \epsilon X - U_\tau$. By passing to a subsequence, we may assume $s(n) \to s \epsilon [0, \tau]$. Then

$$d(T_{s(n)}(x_n), T_s(x)) \leqslant d(T_{s(n)}(x_n), T_s(x_n)) + d(T_s(x_n), T_s(x));$$

arguing as in the proof of Lemma 1 we find that $T_{s(n)}(x_n) \to T_s(x)$. This however contradicts the fact that each $T_{s(n)}(x_n)$ lies in the *closed* set $X - U_\tau$, whereas $x \epsilon U$ implies $T_s(x) \epsilon U_\tau$.

3. Proof of Theorem

The proof follows that in [2], for which the underlying semigroup was that of the nonnegative integers rather than R^+, but the additional details which are needed warrant a full account. The first step is to construct a metric ρ_m, topologically equivalent to d, with respect to which each T_t is *nonexpanding* in the sense that

$$\rho_m(T_t(x), T_t(y)) \leqslant \rho_m(x, y) \qquad (x, y \epsilon X). \tag{3.1}$$

This is accomplished by setting

$$\rho_m(x, y) = \sup \{d(T_t(x), T_t(y)) : t \geq 0\}. \tag{3.2}$$

It will initially be shown that (3.2) yields a finite result. For each $x \epsilon X$, observe that

$$C_x = \{T_\sigma(x) : 0 \leq \sigma \leq \tau\}$$

is a continuous image of $[0, \tau]$, and hence is compact. Let ρ be a metric on X such that T_τ is a contraction on (X, ρ), with $\theta \epsilon (0, 1)$ as contraction constant. Then

$$M_x = \max \{\rho(\xi, z) : z \epsilon C_x\}$$

is finite. Writing any $t \geq 0$ as

$$t = n\tau + \sigma \qquad (\sigma \epsilon [0, \tau];\ \text{integral } n \geq 0), \tag{3.3}$$

we obtain

$$\rho(T_t(x), \xi) = \rho(T_\tau^n T_\sigma(x), T_\tau^n(\xi)) \leq \theta^n \rho(T_\sigma(x), \xi) \leq \theta^n M_x.$$

Since $n \to \infty$ as $t \to \infty$, it follows that

$$T_t(x) \to \xi \qquad \text{as } t \to \infty. \tag{3.4}$$

For particular $x, y \epsilon X$, this implies that

$$d(T_t(x), T_t(y)) \leq d(T_t(x), \xi) + d(\xi, T_t(y)) \to 0$$

as $t \to \infty$. Thus there is a $t_0 > 0$ such that

$$d(T_t(x), T_t(y)) < 1 \qquad \text{for all } t > t_0.$$

Since $d(u, v)$ must be bounded on the compact subset

$$\{(T_t(x), T_t(y)) : 0 \leq t \leq t_0\}$$

of $X \times X$, it follows that (3.2) indeed yields a finite result.

The positive definiteness and symmetry of ρ_m follow directly from the corresponding properties of d. The triangle inequality for ρ_m follows from the fact that, for any $t \geq 0$,

$$d(T_t(x), T_t(y)) \leq d(T_t(x), T_t(z)) + d(T_t(z), T_t(y)) \leq \rho_m(x, z) + \rho_m(z, y).$$

Thus ρ_m is indeed a metric on the set carrying space X. That ρ_m has the nonexpanding property (3.1) follows easily from (3.2). Thus it only remains to prove that ρ_m and d are topologically equivalent. Since $\rho_m \geq d$, we have

$$\rho_m(x_n, x) \to 0 \qquad \text{implies} \qquad d(x_n, x) \to 0,$$

and so it suffices to prove the converse implication.

Consider then any $x \epsilon X$ and any $\eta > 0$. With U as in Lemma 2, there is no loss of generality for what follows in assuming η small enough that

$$S(\xi, \eta) \subseteq U. \tag{3.5}$$

By (3.4), there is a $t(0)>0$ such that

$$d(\xi, T_t(x))<\eta/2 \quad \text{for all} \quad t\geqslant t(0), \tag{3.6}$$

and by Lemma 1 there is a $\delta'>0$ such that

$$T_s(S(x, \delta'))\subseteq S(T_s(x), \eta/2) \quad \text{for all} \quad s\epsilon[0, t(0)]. \tag{3.7}$$

Since $T_t(U)\to\{\xi\}$ as $t\to\infty$, there is a $t(1)>0$ such that

$$T_t(U)\subseteq S(\xi, \eta/2) \qquad \text{for all } t>t(1); \tag{3.8}$$

and by Lemma 1 there is a $\delta''>0$ such that

$$T_s(S(x, \delta''))\subseteq S(T_s(x), \eta/2) \qquad \text{for all } s\epsilon[0, t(0)+t(1)]. \tag{3.9}$$

Let $\delta=\min(\delta', \delta'')$; we will show that

$$T_t(S(x, \delta))\subseteq S(T_t(x), \eta) \qquad (\text{all } t\geqslant 0), \tag{3.10}$$

and establishing the existence (for any x and η) of a δ with this last property completes the current step of the proof.

That (3.10) holds for $0\leqslant t\leqslant t(0)+t(1)$ follows from (3.9). Suppose then that $t>t(0)+t(1)$. By (3.7) with $s=t(0)$, (3.6) with $t=t(0)$, and (3.5), we have

$$T_{t(0)}(S(x, \delta))\subseteq S(\xi, \eta)\subseteq U,$$

and since $t-t(0)>t(1)$, (3.8) yields

$$T_t(S(x, \delta))=T_{t-t(0)}T_{t(0)}(S(x, \delta))\subseteq T_{t-t(0)}(U)\subseteq S(\xi, \eta/2),$$

implying (3.10).

For the *second* step in constructing the desired metric ρ_λ, we begin by observing that the properties (2.3) and (2.4) of U imply

$$T_t(\bar{U})\subseteq\bar{U}; \qquad \overline{T_t(U)}\to\{\xi\} \qquad \text{as } t\to\infty.$$

We now define

$$\nu(x)=\sup\{t: x\epsilon\overline{T_t(U)}\} \qquad \text{for } x\epsilon\bar{U},$$

so that $\nu(\xi)=\infty$ and $0\leqslant\nu(x)<\infty$ for $x\epsilon\bar{U}-\{\xi\}$, and also set

$$\nu(x)=-\inf\{t: x\epsilon T_t^{-1}(\bar{U})\} \qquad \text{for } x\epsilon X-\bar{U},$$

so that $(-\infty)<\nu(x)<0$ in this case.[3] Using the continuity of the semigroup, it is readily shown that the extrema in these definitions of $\nu(x)$ are actually attained. For our purposes, the critical property of $\nu(x)$ is

$$\nu(T_t(x))\geqslant\nu(x)+t, \tag{3.11}$$

[3] The finiteness of $\nu(x)$ is an easy consequence of (3.4).

with verification straightforward.[4] With the further definition

$$c(x, y)=\min \{\nu(x), \nu(y)\} \tag{3.12}$$

it follows from (3.11) that

$$c(T_t(x), T_t(y)) \geqslant c(x, y)+t. \tag{3.13}$$

Now define

$$d_\lambda(x, y)=\lambda^{c(x, y)}\rho_m(x, y), \tag{3.14}$$

which has the correct limiting form $d_\lambda(\xi, \xi)=0$. Then d_λ is positive definite and symmetric, and by (3.13) also has the property

$$d_\lambda(T_t(x), T_t(y)) \leqslant \lambda^t d_\lambda(x, y) \tag{3.15}$$

desired for ρ_λ. However, d_λ may not satisfy the triangle inequality.

This is rectified in the *third* and last step of the construction. Denote by Σ_{xy} the set of chains

$$\sigma_{xy}=[x=x_0, x_1, \ldots, x_m=y]$$

from x to y, with associated *lengths*

$$L_\lambda(\sigma_{xy})=\sum_1^m d_\lambda(x_i, x_{i-1}),$$

and put

$$\rho_\lambda(x, y)=\inf\left\{L_\lambda(\sigma_{xy}):\sigma_{xy}\epsilon\Sigma_{xy}\right\}.$$

We shall show that ρ_λ is the desired metric.

The contraction property

$$\rho_\lambda(T_t(x), T_t(y)) \leqslant \lambda^t\rho_\lambda(x, y)$$

follows by applying (3.15) to the links $[x_{i-1}, x_i]$ of any chain σ_{xy}. Clearly ρ_λ is symmetric and $\rho_\lambda(x, x)=0$; the triangle law holds since following a σ_{xy} with a σ_{yz} yields a σ_{xz}.

It remains to show that ρ_λ is positive definite. For $(-\infty)<\nu\leqslant\infty$, define the closed set

$$K_\nu=\{x\epsilon X:\nu(x)\geqslant\nu\}; \tag{3.16}$$

the sets $\{K_\nu\}$ are a nonascending family, with limit $\{\xi\}$.

Consider any $x\neq\xi$ and any $y\neq x$; assume $\nu(x)\leqslant\nu(y)$ without loss of generality. Choose any $s>0$. First suppose $y\neq\xi$; then $y\epsilon X-K_{\nu(y)+s}$. Thus any chain σ_{xy} either has all its points in $X-K_{\nu(y)+s}$, implying

$$L_\lambda(\sigma_{xy})\geqslant\lambda^{\nu(y)+s}\rho_m(x, y),$$

or else has a last among its links which leaves $K_{\nu(y)+s}$ (and possibly is followed by other links), implying

$$L_\lambda(\sigma_{xy})\geqslant\lambda^{\nu(y)+s}\rho_m(y, K_{\nu(y)+s}).$$

It follows that

$$\rho_\lambda(x, y)\geqslant\lambda^{\nu(y)+s}\min\{\rho_m(x, y), \rho_m(y, K_{\nu(y)+s})\}>0. \tag{3.17}$$

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[4] E.g., for $\nu(x)\geqslant 0$ we have $x\epsilon\overline{T_{\nu(x)}(U)}$, implying $T_t(x)\epsilon T_t(\overline{T_{\nu(x)}(U)})$ and thus $T_t(x)\epsilon\overline{T_tT_{\nu(x)}(U)}=\overline{T_{\nu(x)+t}(U)}$, from which (3.11) follows.

The remaining case, $y = \xi$, is covered by applying analogous reasoning to yield

$$\rho_\lambda(x, \xi) \geqslant \lambda^{\nu(x)+s}\rho_m(x, K_{\nu(x)+s}) > 0. \tag{3.18}$$

By (3.17) and (3.18), ρ_λ is positive definite and hence indeed a metric, which must still be proved equivalent to ρ_m.

Now define

$$\mu(x) = \inf\{\mu \geqslant 0 : T_\mu(x) \epsilon U\} < \infty; \tag{3.19}$$

it follows from the definition of $\nu(x)$ that

$$\nu(x) \geqslant -\mu(x). \tag{3.20}$$

With the further definition

$$B_\mu = X - T_\mu^{-1}(U) \tag{3.21}$$

we have $\rho_m(x, B_{\mu(x)+s}) > 0 \qquad$ for all $s > 0$.

Consider any $x \epsilon X$; we will show that

$$\rho_m(x_n, x) \to 0 \qquad \text{implies} \qquad \rho_\lambda(x_n, x) \to 0. \tag{3.22}$$

First suppose $x \neq \xi$; choose any $s > 0$. Consider any $y \epsilon X$ with

$$\rho_m(x, y) < \rho_m(x, B_{\mu(x)+s}). \tag{3.23}$$

Then $y \epsilon X - B_{\mu(x)+s}$, implying $T_{\mu(x)+s}(y) \epsilon U$ and hence that $\mu(x) + s \geqslant \mu(y)$. By (3.20), this implies $\nu(y) \geqslant -\mu(x) - s$. Since (3.20) also implies $\nu(x) \geqslant -\mu(x) - s$, we have $c(x, y) \geqslant -\mu(x) - s$, so that (3.23) implies the second inequality in

$$\rho_\lambda(x, y) \leqslant d_\lambda(x, y) \leqslant \lambda^{-\mu(x)-s}\rho_m(x, y). \tag{3.24}$$

Since (3.23) implies the indicated inequality between the first and third terms of (3.24), (3.22) has been established for all $x \neq \xi$. As for the case $x = \xi$, note that if $\rho_m(y, \xi) < \rho_m(\xi, B_0)$, then $\nu(y) \geqslant 0$, and so $c(\xi, y) \geqslant 0$ and

$$\rho_\lambda(\xi, y) \leqslant d_\lambda(\xi, y) \leqslant \rho_m(\xi, y), \tag{3.25}$$

showing that (3.22) holds in this case also.

It only remains to prove that

$$\rho_\lambda(x_n, x) \to 0 \qquad \text{implies} \qquad \rho_m(x_n, x) \to 0. \tag{3.26}$$

As above, first assume $x \neq \xi$, and choose any $s > 0$. Since $K_\nu \to \{\xi\}$, the argument culminating in (3.25) shows that there exists $k(x) > \max\{0, \nu(x) + s\}$ such that $z \epsilon K_{k(x)}$ implies $\rho_\lambda(\xi, z) < \rho_\lambda(x, \xi)/2$. Then

$$\rho_\lambda(x, K_{k(x)}) \geqslant \rho_\lambda(x, \xi)/2. \tag{3.27}$$

Consider any $y \epsilon X$ for which

$$\rho_\lambda(x, y) < \rho_\lambda(x, \xi)/2. \tag{3.28}$$

It follows from (3.27) that in forming $\rho_\lambda(x, y)$ as an infimum, only chains disjoint from $K_{k(x)}$ need be considered, implying

$$\rho_\lambda(x, y) \geqslant \lambda^{k(x)}\rho_m(x, y). \tag{3.29}$$

Now suppose that y, besides obeying (3.28), also satisfies

$$\rho_\lambda(x, y) < \lambda^{k(x)}\rho_m(x, K_{\nu(x)+s}). \tag{3.30}$$

Then (3.29–30) imply

$$\rho_m(x, y) < \rho_m(x, K_{\nu(x)+s}). \tag{3.31}$$

We will show that

$$\rho_m(x, y) \leqslant \lambda^{-\nu(x)-s}\rho_\lambda(x, y); \tag{3.32}$$

that this is implied by (3.28) and (3.30), shows that (3.26) indeed holds for $x \neq \xi$.

In case $\nu(y) \leqslant \nu(x)$, (3.32) is obtained by applying (3.17) with x and y reversed, and then taking account of (3.31). In the contrary case, (3.31) implies $y \epsilon X - K_{\nu(x)+s}$; thus any chain σ_{xy} either lies in $X - K_{\nu(x)+s}$, implying

$$L_\lambda(\sigma_{xy}) \geqslant \lambda^{\nu(x)+s}\rho_m(x, y),$$

or else has a first among its links which enters $K_{\nu(x)+s}$, implying

$$L_\lambda(\sigma_{xy}) \geqslant \lambda^{\nu(x)+s}\rho_m(x, K_{\nu(x)+s}).$$

Taking (3.31) into account, it follows that

$$\rho_\lambda(x, y) \geqslant \lambda^{\nu(x)+s}\rho_m(x, y),$$

which is equivalent to the desired result (3.32).

Finally, (3.26) must be verified in the case $x = \xi$. Note that for any $\eta > 0$, there is a $\nu > 0$ such that $\rho_m(\xi, z) < \eta/2$ for all $z \epsilon K_\nu$. Consider any $y \epsilon X$ such that

$$\rho_m(\xi, y) > \eta; \tag{3.33}$$

it follows that $\nu(y) < \nu$ and $\rho_m(y, K_\nu) \geqslant \eta/2$. Choose $s > 0$ so that $\nu(y) + s = \nu$; then (3.18) yields the first inequality in

$$\begin{aligned} \rho_\lambda(\xi, y) &\geqslant \lambda^\nu \rho_m(y, K_{\nu(y)+s}) \\ &= \lambda^\nu \rho_m(y, K_\nu) \geqslant \lambda^\nu \eta/2 \equiv \delta > 0. \end{aligned} \tag{3.34}$$

The contrapositive, of the implication of (3.34) by (3.33), is that $\rho_\lambda(\xi, y) < \delta$ implies $\rho_m(\xi, y) \leqslant \eta$. Since for each $\eta > 0$ there is a $\delta > 0$ with this property, (3.26) has indeed been verified for $x = \xi$.

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