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On the Singular Values of a Product of Matrices*

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The purpose of this note is to give necessary and sufficient conditions for the singular values of a product of matrices to be equal to certain products of their singular values. We then analyze the case of equality in a matrix inequality of Ostrowski.

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The purpose of this note is to give necessary and sufficient conditions for the singular values of a product of matrices to be equal to certain products of their singular values. We then analyze the case of equality in a matrix inequality of Ostrowski.

The singular values of an n-square complex matrix X are the positive square roots of the eigenvalues of X^*X, where X^* is the conjugate transpose of X. Denote the singular values of X by $\alpha_1(X), \ldots, \alpha_n(X)$, arranged so that $\alpha_1(X) \geqslant \ldots \geqslant \alpha_n(X) > 0$ (all matrices are assumed to be nonsingular). Let A and B be n-square complex matrices and let $A=UH$, $B=VK$ be the polar factorizations of A and B. In the factorizations U and V are unitary matrices and H and K are positive-definite hermitian matrices.

THEOREM 1: *Let* k *be a positive integer less than* n. *Then*

$$\alpha_i(AB)=\alpha_i(A)\alpha_i(B), \qquad \textit{for} \qquad 1 \leqslant i \leqslant k \tag{1}$$

if and only if there exists a unitary matrix W *such that*

$$W^*V^*HVW = \textit{diag}\ (\alpha_1(A), \ldots, \alpha_k(A)) \dot{+} T_1. \tag{2}$$

and

$$W^*KW = \textit{diag}\ (\alpha_1(B), \ldots, \alpha_k(B) \dot{+} T_2$$

where T_1 *and* T_2 are $(n-k)$*-square matrices.*

PROOF: Since the singular values of AB are the same as the singular values of $(V^*HV)K$, it suffices to prove this theorem for the case where A and B are positive-definite hermitian matrices.

If there is a unitary matrix W satisfying

$$W^*AW = \text{diag}\ (\alpha_1(A), \ldots, \alpha_k(A)) \dot{+} T_1$$

and

$$W^*BW = \text{diag}\ (\alpha_1(B), \ldots, \alpha_k(B)) \dot{+} T_2 \tag{3}$$

then $\qquad \alpha_i(AB) = \alpha_i(A)\alpha_i(B) \qquad 1 \leqslant i \leqslant k.$

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We use an induction argument on the size of A and B, n, to show that condition (1) implies there is a unitary matrix W satisfying (3). Let $|v| = (\Sigma_1 |v_i^2|)^{\frac{1}{2}}$ denote the length of the n-tuple $v = (v_1, \ldots, v_n)$.

If $n=1$, then $W=[1]$ will satisfy (3).

If $n \geqslant 2$, let y be an n-tuple of unit length such that

$$\alpha_1(AB) = |ABy| = |By| \cdot \left| A\left(\frac{By}{|By|}\right)\right| \cdot \tag{4}$$

But $|By| \leqslant \alpha_1(B)$ and $\left| A\left(\frac{By}{|By|}\right)\right| \leqslant \alpha_1(A)$. By hypothesis, $\alpha_1(AB) = \alpha_1(A)\alpha_1(B)$, so $|By| = \alpha_1(B)$. Since B is positive-definite hermitian. $By = \alpha_1(B)y$. Also $Ay = \alpha_1(A)y$.

Let S be a unitary matrix whose first column is y. Since S^*AS is hermitian, $S^*AS = \alpha_1(A) \dotplus A'$, where A' is an $(n-1)$-square, positive-definite hermitian matrix. Similarly, $S^*BS = \alpha_1(A) \dotplus B'$ where B' is an $(n-1)$-square positive-definite hermitian matrix. Clearly,

$$\alpha_i(A') = \alpha_{i+1}(A) \qquad 1 \leqslant i \leqslant n-1,$$

$$\alpha_i(B') = \alpha_{i+1}(B) \qquad 1 \leqslant i \leqslant n-1,$$

and

$$\alpha_i(A'B') = \alpha_{i+1}(AB) \qquad 1 \leqslant i \leqslant n-1.$$

So equality (1) implies that

$$\alpha_i(A'B') = \alpha_i(A')\alpha_i(B') \qquad 1 \leqslant i \leqslant k-1.$$

By the induction hypothesis applied to the $(n-1)$-square matrices A' and B', there exists an $(n-1)$-square unitary matrix S' such that

$$S'^*A'S' = \operatorname{diag}(\alpha_2(A), \ldots, \alpha_k(A)) \dotplus T_1$$

and

$$S'^*B'S' = \operatorname{diag}(\alpha_2(A), \ldots, \alpha_k(A)) \dotplus T_2$$

where T_1 and T_2 are $(n-k)$-square matrices.

Finally let $W = S(1 \dotplus S')$, then

$$\begin{aligned} W^*AW &= (1 \dotplus S'^*)S^*AS(1 \dotplus S') \\ &= (1 \dotplus S'^*)(\alpha_1(A) \dotplus A')(1 \dotplus S') \\ &= \alpha_1(A) \dotplus S'^*A'S' \\ &= \operatorname{diag}(\alpha_1(A), \ldots, \alpha_k(A)) \dotplus T_1. \end{aligned}$$

Similary $W^*BW = \operatorname{diag}(\alpha_1(B), \ldots, \alpha_k(B)) \dotplus T_2$. *Q.E.D.*

We need the following definition in order to state Ostrowski's inequality.

DEFINITION: *Let* $\phi(x_1, \ldots, x_k)$ *be a real valued function of* k *real variables.* ϕ *is* ***convex*** *in a region* R *if*

$$\phi(\theta x + (1-\theta) y) \leqslant \theta\phi(x) + (1-\theta)\phi(y) \tag{5}$$

whenever $0 < \theta < 1$, $x = (x_1, \ldots, x_k) \epsilon R$, $y = (y_1, \ldots, y_k) \epsilon R$. *If equality holds in* (5) *only when* $x_i = y_i$, $1 \leqslant i \leqslant k$, *we say* ϕ *is* ***strictly convex***.

Now we state two theorems by Ostrowski [3, Thm. XVI].

THEOREM 2 (Ostrowski): *Let* $f(x_1, \ldots, x_k)$ *be a symmetric function of* k *real variables such that*

$$\phi(x_1, \ldots, x_k) = f(\exp x_1, \ldots, \exp x_k)$$

is increasing in each variable x_i *and convex in the region* $x_i \geqslant 0$. *Then,*

$$f(\alpha_1(AB), \ldots, \alpha_k(AB)) \leqslant f(\alpha_1(A)\alpha_1(B), \ldots, \alpha_k(A)\alpha_k(B)). \tag{6}$$

THEOREM 3 (Ostrowski): *Suppose* $\phi(x_1, \ldots, x_k)$ *is a symmetric function which is convex and increasing in each variable. Let* $\{x_i, y_i\}$ *be* 2n *positive numbers satisfying*

$$\begin{gathered} x_1 \geqslant \ldots \geqslant x_n, \qquad y_1 \geqslant \ldots \geqslant y_n \\ x_1 + \ldots + x_r \leqslant y_1 + \ldots + y_r; \qquad 1 \leqslant r \leqslant n, \end{gathered} \tag{7}$$

with equality in (7) *for* $r = n$. *Then*

$$\phi(x_1, \ldots, x_k) \leqslant \phi(y_1, \ldots, y_k). \tag{8}$$

In addition, if ϕ *is strictly convex and strictly increasing in each variable, then equality holds in* (8) *if and only if* $x_i = y_i$, $1 \leqslant i \leqslant k$.

If the inequalities in (7) hold, x is said to *majorize* y, [1, pg. 45]. A proof of Theorem 3 different than Ostrowski's is given in [2]. Theorem 2 follows from Theorem 3 by choosing $x_i = \log \alpha_i(AB)$ and $y_i = \log(\alpha_i(A)\alpha_i(B))$. It is known that

$$\prod_{i=1}^{r} \alpha_i(AB) \leqslant \prod_{i=1}^{r} \alpha_i(A)\alpha_i(B), \qquad 1 \leqslant r \leqslant n.$$

Now using Theorem 1, it is easy to show that if ϕ is strictly convex and strictly increasing in each variable, then equality holds in Theorem 2, (6) if and only if there exists a unitary matrix W such that (2) holds.

References

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[2] Marcus, Marvin, and Gordon, William R., Analysis of equality in certain matrix inequalities (submitted).

[3] Ostrowski, A., Sur quelques applications des fonctions convexes et concaves au sens de I. Schur, Mathématiques Pures et Appliquees, **IV,** 253–292 (1952).

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