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Some Theorems on Tensor Composite Graphs*

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If a graph (digraph) is isomorphic to the tensor product of two graphs (digraphs) it is said to be a tensor composite graph (digraph). If not, it is said to be tensor prime. Several theorems giving various properties of tensor composite graphs and digraphs are presented. Among those dealing with (undirected) graphs is the result that any tree is tensor prime. This does not hold for digraphs. An example is given of a tensor composite digraph which is an unoriented tree. It is proved that a tensor composite digraph which is an oriented tree (an arborescence) does not exist. Some applications are presented.

Key words: Digraphs; graphs; products; tensor; trees.

1. Introduction

We refer to Ore [8],[1] Berge [1], and Harary et al. [4], for the usual definitions of graph, digraph (directed graph) and related terms. The tensor product of two graphs G_u and G_v is denoted by $G_u \otimes G_v$ and defined as follows: If V_u and V_v are the sets of vertices of G_u and G_v respectively, then the set of vertices of $G_u \otimes G_v$ is $V_u \times V_v$. Two vertices (u_1, v_1), (u_2, v_2) of $G_u \otimes G_v$ are adjacent if and only if u_1 and u_2 are adjacent in G_u, *and* v_1 and v_2 are adjacent in G_v. If G_u and G_v are digraphs then (u_1, v_1) is adjacent to (u_2, v_2) if and only if u_1 is adjacent to u_2 and v_1 is adjacent to v_2.

A number of papers, [2, 5, 6, 9], have studied various problems of characterizing the component graphs G_u and G_v so that $G_u \otimes G_v$ has certain properties. In this paper, as in an earlier one [3], we focus our attention on the structure of any graph which is isomorphic to the tensor product of two graphs, and investigate its properties. Such a graph (digraph) is called *tensor composite.* If there do not exist graphs G_u and G_v such that $G \cong G_u \otimes G_v$ then G is said to be *tensor prime.*

2. Main Theorems

Our first theorem lists a number of necessary conditions for a graph to be tensor composite. We require the following definition of the distance between two lines of a graph. Let l_1 be the line joining u and v (written $u \sim v$), and let l_2 be $w \sim x$. The *distance* between l_1 and l_2, denoted by $d(l_1, l_2)$, is the length of the shortest path joining any vertex of l_1 with any vertex of l_2.

THEOREM 1: *Let* G *be a tensor composite graph with* n *vertices and* l *lines* ($l > 0$) then (1) $\mathrm{n} = \mathrm{n_u n_v}$ *where* $\mathrm{n_u}$ *and* $\mathrm{n_v}$ *are integers greater than* 1 *such that for all vertices* v *of* G, $\mathrm{d(v)} \leq (\mathrm{n_u}-1)(\mathrm{n_v}-1)$ *where* d(v) *is the degree of* v, (2) $l \leq 2\binom{\mathrm{n_u}}{2}\binom{\mathrm{n_v}}{2}$ (3) $l = 2\mathrm{k}$, for some $\mathrm{k} = 1, 2, \ldots\ldots, \binom{\mathrm{n_u}}{2}\binom{\mathrm{n_v}}{2}$ *and these* l *lines can be listed in* k *pairs such that for each such pair* l_1, l_2, $\mathrm{d}(l_1, l_2) > 1$.

PROOF: (1) Let G be isomorphic to $G_u \otimes G_v$ and let n_u and n_v be the number of vertices of G_u and G_v, respectively. Then $n_u > 1$, $n_v > 1$, for otherwise $l = 0$, and furthermore it is clear that $n_u n_v = n$. For the vertices of $G_u \otimes G_v$ we use the abbreviation $(u_i, v_j) = u_i v_j$. Now since $u_i \not\sim u_i$ and $v_i \not\sim v_i$ for any i,

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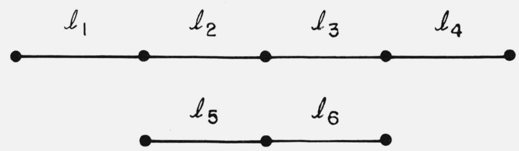
[1] Figures in brackets indicate the literature references at the end of this paper.

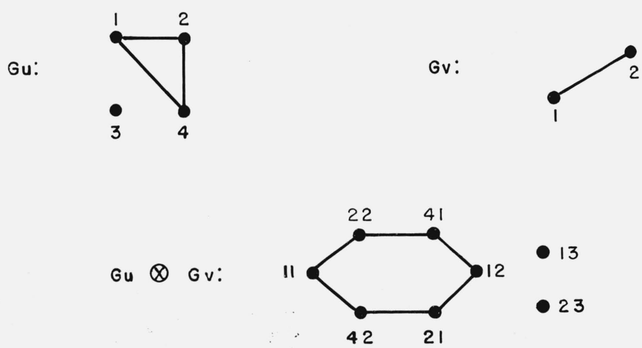
u_iv_j is adjacent with at most $(n_u-1)(n_v-1)$ vertices. Hence $d(v)\leqslant(n_u-1)(n_v-1)$ for all v in G. (2) $l=\frac{1}{2}\sum_{v\in G} d(v)\leqslant\frac{1}{2}\sum_{v\in G}(n_u-1)(n_v-1)\leqslant\frac{1}{2}n_un_v(n_u-1)(n_v-1)=2\binom{n_u}{2}\binom{n_v}{2}$. (3) Suppose $u_1v_1\sim u_2v_2$. Then $u_1v_2\sim u_2v_1$. Also $u_1v_1\nsim u_1v_2$, $u_1v_1\nsim u_2v_1$, $u_2v_2\nsim u_1v_2$, $u_2v_2\nsim u_2v_1$. Hence the distance between these two lines is greater than 1.

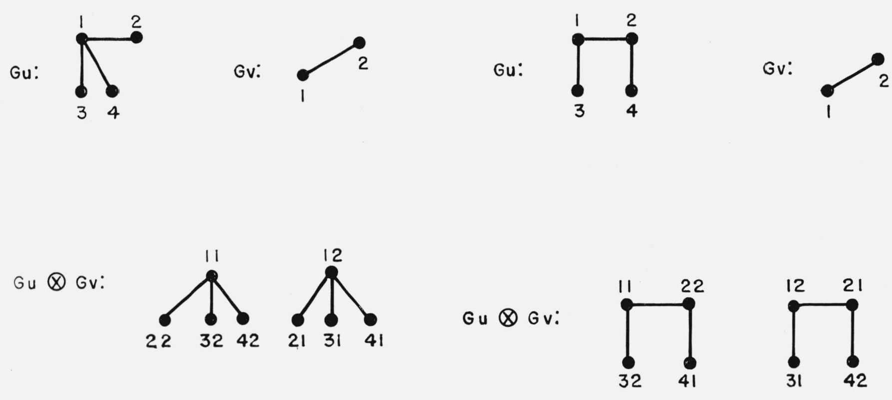
The above theorem has a parallel for digraphs with a similar proof [3].

Note that if l is the number of lines in a tensor composite graph and l_u and l_v are the numbers of lines in the two components graphs then $l=2l_ul_v$. We use this fact in an example which shows that the conditions in theorem 1 are not sufficient, Let G be the graph shown below.



Then $n=8=4\times 2$, $d(v)\leqslant 2\leqslant 3$, $l=6$, $d(l_1,\ l_4)>1$, $d(l_2,\ l_6)>1$, $d(l_3,\ l_5)>1$. However, for G to be tensor composite G_u, say, would have to have 4 vertices and 3 lines, and G_v, 2 vertices and 1 line. The only possibilities are shown below, and G is none of these.





The problem of characterizing tensor composite graphs and digraphs seems quite complex and remains unsolved. A characterization of digraphs having a prime number of lines was given in [3]. The remainder of this paper will be devoted to two nonexistence theorems and their applications.

THEOREM 2: *No tree is tensor composite.*

PROOF: We use two lemmas.

LEMMA 1: *If* G_u *or* G_v *has a cycle, then so does* $G_u\otimes G_v$.

PROOF: Let

$$u_1 \sim u_2 \sim u_3 \sim \ldots, \sim u_k \sim u_1,\ v_1 \sim v_2.$$

Then if k is even,

$$u_1v_1 \sim u_2v_2 \sim u_3v_1 \sim u_4v_2 \sim v_5v_1 \sim \ldots \sim u_kv_2 \sim u_1v_1$$

so that $G_u \otimes G_v$ has a cycle. If k is odd then

$$u_1v_1 \sim u_2v_2 \sim u_3v_1 \sim u_4v_2 \sim \ldots \sim u_kv_1 \sim u_1v_2 \sim u_2v_1 \sim \ldots \sim u_kv_2 \sim u_1v_1$$

so that $G_u \otimes G_v$ again has a cycle.

LEMMA 2: *The tensor product of two forests is not connected.*

PROOF: This lemma follows, in part, from Weichel's theorem 1, [9], which states that a tensor composite graph is connected if and only if the component graphs are connected and at least one of them has an odd cycle. For the sake of completeness, we give an independent proof of the lemma here.

Consider u_1v_1 and u_1v_2 with $v_1 \sim v_2$. We claim that there is no path between u_1v_1 and u_1v_2. There can not be a path between u_1 and u_1 since this is a cycle and its existence would contradict the fact that G_u is a forest. Therefore, the only way one can have a path between u_1v_1 and u_1v_2 is if this path has the form

$$u_1v_1 \sim u_2v_j \sim u_1v_k \sim u_2v_m \sim \ldots . \sim u_1v_2.$$

We examine the possibilities. Of course $j \neq 1$, $k \neq 1$. If $j=2$ then $k \neq 2$. Let $k=3$. Then $m \neq 1$ for otherwise there would exist a cycle containing v_1. It is also clear that $m \neq 2$. Therefore, $m=4$. But this implies the existence of a cycle containing v_2. Hence we must have $j=3$. However, this implies the existence of a path between v_1 and v_2 other than $v_1 \sim v_2$, which would mean that v_1 and v_2 are contained in a cycle. This final contradiction proves lemma 2.

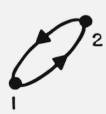
Since a tree is a connected graph without cycles, the theorem follows immediately.

Lemma 2 has a parallel for digraphs. One talks about semicycles instead of cycles, and weakly connected instead of connected. Thus if we understand that a forest is a digraph without semicycles than we can prove the following result.

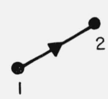
LEMMA 2′: *The tensor product of two forests is not weakly connected.*

Lemma 1, however does not hold for digraphs, as can be seen by the following example.

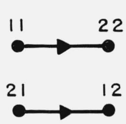
Du:



Dv:



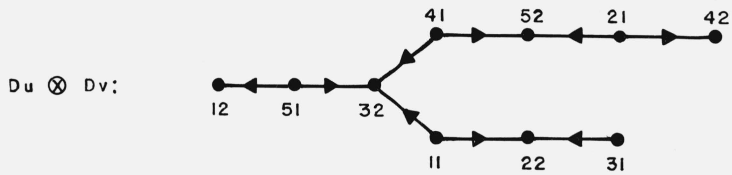
Du ⊗ Dv:



Hence the proof of theorem 2 does not go through for digraphs. In fact, the theorem is not true, as can be seen by the example below.

Note that in this example the tree obtained is an unoriented one. One immediately asks if an oriented tree (an arborescence) can be tensor composite. The answer is given in the next theorem.
THEOREM 3: *Except for the trivial graph (a single point), any arborescence is tensor prime.*
PROOF: Consider a nontrivial tensor composite digraph D in which u_1v_1 has indegree 0. Then $u_iv_j \nrightarrow u_1v_1$ for all i and j. Therefore, either $u_i \nrightarrow u_1$ for all i or $v_j \nrightarrow v_1$ for all j. Hence for some $k \neq 1$ either u_1v_k or u_kv_1 has indegree 0. Since an arborescence requires exactly one point with *indegree* 0, D can not be an arborescence.

Using a dual argument it is easy to see that no nontrivial tensor composite digraph can have exactly one point with *outdegree* 0.

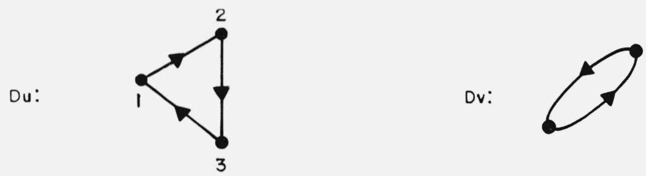


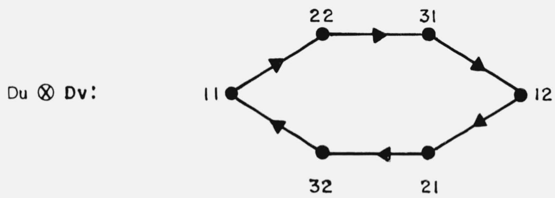
3. Some Applications

We present applications of theorems 2 and 3. Theorem 2 is used in the form: Any connected tensor composite graph has at least one cycle.

Consider a message which for security purposes is to be transmitted in two parts, possibly a text and a key. We have a group of n agents among whom part 1 can be transmitted, and a group of m agents among whom part 2 can be transmitted. We represent these two groups by two graphs in which the vertices are the agents and two vertices are adjacent if and only if the two agents they represent communicate with each other. The tensor product of these two graphs then represents the communication pattern for the entire message among all possible "teams" of two agents each. Theorem 2 tells us that if things are arranged in such a way that the message can get from any agent team to any other (the tensor product is connected) then there must exist at least three teams which can experience feedback in the sense that a message can be returned to them (the tensor product has at least one cycle).

Theorem 3 can be applied to a situation in which two sets of players compete in tournaments. We represent each set by a digraph in which the vertices are the players and u is adjacent to v if and only if u defeats v. In the case of a tie we let u and v be adjacent to each other. Now form all possible teams, or coalitions, of two players, one from the first set and one from the second set. Assuming that the strengths of the players are additive, the tensor product will represent the "victory-defeat" pattern for these coalitions. From the proof of theorem 3 we see that there can not be exactly one undefeated coalition. Either there is none, or there are more than one. That the latter is possible can be seen by our last two examples. The former is illustrated below.





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