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A Property of the Triangle Groups

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The F-groups are the groups possessing faithful representations by Fuchsian groups of the first kind; their presentations are known explicitly. Among the F-groups are the well-known triangle groups $G=\{x, y|x^p=y^q=(xy)^r=1\}$. If p, q, r are distinct prime integers, every proper normal subgroup of finite index in G has no elements of finite order. In this paper it is proved that among the F-groups only the triangle groups with distinct prime p, q, r have this property.

Key words: Element of finite order; F-group; normal subgroup; triangle group.

1. The F-groups of J. Nielsen consist of those groups that can be faithfully represented by Fuchsian groups of the first kind. Such groups are known to be exactly those that have presentations of the form

$$\Big\{e_1, \ldots, e_s, p_1, \ldots, p_t, a_1, b_1, \ldots, a_g, b_g \mid e_1^{l_1} = \ldots = e_s^{l_s}$$

$$= \prod_{i=1}^{s} e_i \prod_{j=1}^{t} p_j \prod_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1} = 1 \Big\} \tag{1.1}$$

where $s, t, g \geqslant 0$, $l_i \geqslant 2$ and

$$g-1+\frac{1}{2}\left(t+\sum_{i=1}^{s}\left(1-\frac{1}{l_i}\right)\right)>0. \tag{1.2}$$

For each such group G it has been proved that there exists a proper normal subgroup of finite index with no elements of finite order [1, 2].[1]

On the other hand, the particular F-group

$$\Delta=\{e_1, e_2, e_3 | e_1^2=e_2^3=e_3^7=e_1e_2e_3=1\}$$

has the property that *every* proper normal subgroup of finite index is torsion free.

We shall say that a group satisfying this stronger condition has property A. Since a torsion-free group has the property trivially, we make the following definition. A group has property A if it has elements of finite order and if every proper normal subgroup of finite index is torsion-free.

The question arises: which F-groups have property A?

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[1] Figures in brackets indicate the literature references at the end of this paper.

THEOREM: *Let G be an F-group. Necessary and sufficient that G have property A is that G be a triangle group with distinct prime exponents. Explicitly, this means that in (1.1) we have*

$$g=t=0, \qquad s=3$$

and the $\{l_i\}$ *are distinct primes.*

The presentation of the triangle group can of course be written more compactly as

$$\{e_1, e_2 | e_1^{l_1} = e_2^{l_2} = (e_1 e_2)^{l_3} = 1\}\cdot$$

2. For purposes of this paper we define an F-group as one that has a presentation (1.1) with condition (1.2) satisfied. The following facts are well known. Every F-group G has a faithful representation by a Fuchsian group of the first kind $\bar{G}$:

$$\bar{G} = \Big\{ E_1, \ldots, E_s, P_1, \ldots, P_t, A_1, B_1, \ldots, A_g, B_g | E_1^{l_1} = \ldots$$

$$= E_s^{l_s} = \prod_i E_i \prod_j P_j \prod_k A_k B_k A_k^{-1} B_k^{-1} = 1 \Big\}, \tag{2.1}$$

in which E_i is an elliptic, P_j is a parabolic, and A_k, B_k are hyperbolic elements of $SL(2, R)$. Every element in $\bar{G}$ of finite order has order dividing an l_i and is conjugate over $\bar{G}$ to a nontrivial power of E_i. If $\bar{G}$ has an element of finite order, this element is elliptic and has a fixed point lying in the upper half-plane which is a vertex of a fundamental region of $\bar{G}$. This implies that G has an elliptic generator, so that if $\bar{G}$ has an element of finite order, then $s > 0$.

3. LEMMA 1: *If* G *has property* A, *then* $g=t=0$ *and* $s \geqslant 3$ *in (2.1)*

Since G has elements of finite order, $s > 0$. Suppose $g > 0$. Let N be the kernel of the homomorphism $G \to Z_2$ sending $a_1 \to \theta$ and all other generators to 1, where θ is a generator of Z_2, the cyclic group of order 2. Then $[G{:}N] = 2$ and N contains e_1, an element of finite order l_1. Hence G does not have property A.

Therefore $g=0$. By (1.2) we have

$$s+t \geqslant 3.$$

Suppose $t > 0$. G is isomorphic to a free product G_1, since p_1 may be eliminated:

$$G \simeq G_1 = \{e_1, \ldots, e_s, p_2, \ldots, p_t | e_1^{l_1} = \ldots = \ldots e_s^{l_s} = 1\}.$$

If $t \geqslant 2$, $p_t \epsilon G_1$. Map $p_t \to \theta$, a generator of Z_2, and all other generators of G_1 to 1; the kernel of this map contains e_1. If $t=1$, we must have $s \geqslant 2$; G is then the free product of s cyclic groups of finite order. In that case map $e_1 \to \theta$, a generator of Z_{l_1}, and all other generators of G_1 to 1. The kernel of this map contains e_2.

This completes the proof. G is now of the form

$$G = \}e_1, \ldots, e_s | e_1^{l_1} = \ldots = e_s^{l_s} = e_1 \ldots e_s = 1\}, \ s \geqslant 3. \tag{3.1}$$

LEMMA 2: *If* G *has property* A, *the exponents* l_i *in (3.1) are distinct primes, and* $s=3$.

Suppose, say, $l_1 = mn$, $m > 1$, $n > 1$. Let N be the normal closure in G of e_1^m and let $H = G/N$. A presentation for H is

$$\{e_1, \ldots, e_s | e_1^m = e_2^{l_2} = \ldots = e_s^{l_s} = e_1 \ldots e_s = 1\}.$$

Since $s > 2$, $H \neq \{1\}$. By making use of the theorem of Fox (see [2]), we can deduce the existance of a proper normal subgroup H_1 of H of finite index. If we denote by G_1 the inverse image of H_1 by canonical homomorphism, then G_1 is a proper normal subgroup in G of finite index and G_1 contains e_1^m, an element of finite order n. Hence G does not have property A.

It follows that all l_i are prime. Recall from Lemma 1 that there are at least 3 generators e_i. If $l_1 = l_2$, say, map $e_1 \to \theta$, $e_2 \to \theta^{-1}$, and all other generators to 1, where θ is a generator of $Z_{l_1} = Z_{l_2}$. The kernel of this map contains e_1, $i \neq 1, 2$. Hence the $\{l_i\}$ are distinct.

Finally, suppose $s > 3$. Let N be the normal closure of e_1 in G; let $H = G/N$. H has the presentation $\{e_2, \ldots, e_s | e_2^{l_2} = \ldots = e_s^{l_s} = e_2 \ldots e_s = 1\}$. As before $H \neq \{1\}$, the number of independent generators being $s-2 > 1$, and has elements of finite order. Reasoning as above, we conclude the existence of a proper normal subgroup G_1 of finite index in G and G_1 contains e_1, a contradiction.

Lemmas 1 and 2 complete the necessity part of the Theorem. To prove the sufficiency, consider

$$G = \{x, y | x^p = y^q = (xy)^r = 1\}$$

with distinct primes p, q, r. Let N be any normal subgroup of G that contains an element of finite order, u. Then u is conjugate to some nontrivial power of x, y, or xy, say x^α, $0 < \alpha < p$, and since N is normal, x^α itself lies in N. Since p is a prime, $x \epsilon N$. Now G/N is generated by the images $\bar{x}$, $\bar{y}$ of x, y under the homomorphism $G \to G/N$. Then $\bar{x} = 1$ and so $\bar{y}^q = (\bar{x}\bar{y})^r = \bar{y}^r = 1$, implying $\bar{y} = 1$ since $(q, r) = 1$. Hence $G/N = \{1\}$ or $N = G$. This completes the proof.

References

[1] Bundgard, S., and Nielsen, J., On normal subgroups with finite index in F-groups, Matematisk Tidsskrift **B**, 56–58 (1951).

[2] Fox, R. H., On Fenchel's conjecture about F-groups, Matematisk Tidsskrift **B**, 61–65 (1952).

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