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On the Spheroidal Functions *

Donald R. Rhodes **

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A number of new properties of the spheroidal functions of arbitrary real order $\alpha > -1$ are established, including double orthogonality over two separate intervals simultaneously and the existence of a new kind of characteristic numers $\gamma_{\alpha n}(c)$ that arise from it. Some computational formulas are derived and a few numerical results are shown.

Key words: Mathieu functions; spheroidal functions.

1. Introduction

The spheroidal functions as defined and investigated by Stratton [1] [1] and later by Chu and Stratton [2] are those solutions of the differential equation

$$(1-\eta^2)\psi''_{\alpha n}(c, \eta) - 2(\alpha+1)\eta\psi'_{\alpha n}(c, \eta) + (b_{\alpha n} - c^2\eta^2)\psi_{\alpha n}(c, \eta) = 0 \tag{1}$$

that remain finite at the singular points $\eta = \pm 1$. The condition of finiteness restricts the admissible values of the parameter $b_{\alpha n}(c)$ to a discrete set of eigenvalues, indexed by $n = 0, 1, 2, 3, \ldots$, that depend upon the values chosen for the order α and the parameter c. The differential equation (1) together with the condition of finiteness at $\eta = \pm 1$ is equivalent to the integral equation [1(27)]

$$\nu_{\alpha n}(c)\psi_{\alpha n}(c, \eta) = \int_{-1}^{1} e^{ic\eta t}(1-t^2)^{\alpha}\psi_{\alpha n}(c, t)dt, \quad -1 \leq \eta \leq 1, \tag{2}$$

with eigenvalues $\nu_{\alpha n}(c)$, valid for real $\alpha > -1$.

For integral orders $\alpha = m = 0, 1, 2, 3, \ldots$ the functions $\psi_{\alpha n}(c, \eta)$ are related to the spheroidal wave functions $S_{m, m+n}(c, \eta)$ that arise from separation of the wave equation in spheroidal coordinates. For half-integral orders $\alpha = \pm\frac{1}{2}$ they are related to the periodic Mathieu functions $Se_n(c, \eta)$ and $So_{n+1}(c, \eta)$ that arise from separation of the wave equation in elliptic cylinder coordinates. The purpose of the present paper is to derive some new properties of the spheroidal functions of arbitrary real order $\alpha > -1$, including a new kind of characteristic numbers $\gamma_{\alpha n}(c)$ that arise from the property of double orthogonality shown here to be possessed by these functions, and to present some numerical results illustrating them. It will be assumed throughout that the parameter c is real and positive, although many of the results are more general. A secondary purpose is to emphasize the underlying unity of the properties associated with the spheroidal functions of arbitrary real order $\alpha > -1$. All properties of the spheroidal wave functions and of the periodic Mathieu functions arise quite naturally from those of the general functions $\psi_{\alpha n}(c, \eta)$. An extensive and completely general treatment for all solutions of (1) for arbitrary complex values of α, b, and c has been developed by Meixner and Schäfke [3].

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** Present address: Department of Electrical Engineering, North Carolina State University, Raleigh, North Carolina 27607.

[1] Figures in brackets indicate the literature references at the end of this paper.

2. Elementary Properties

The following properties of the spheroidal functions $\psi_{\alpha n}(c,\eta)$ are evident directly from either the differential equation (1) and its boundary condition of finiteness at $\eta=\pm 1$ or the equivalent integral equation (2).

(a) The functions $\psi_{\alpha n}(c,\eta)$ are entire functions of η. This follows from the fact that the parameter $b_{\alpha n}$ was chosen in such a way that one of the two solutions of differential equation (1) remains finite at the singular points $\eta=\pm 1$. With no singularities except for the one at infinity that solution $\psi_{\alpha n}(c,\eta)$ must be entire.

(b) As $c\to 0$ the functions $\psi_{\alpha n}(c,\eta)$ become proportional to the Gegenbauer functions $T_n^\alpha(\eta)$, and the eigenvalues $b_{\alpha n}(c)$ approach

$$b_{\alpha n}(c)\to n(n+2\alpha+1). \tag{3}$$

This follows from the fact that the differential equation (1) approaches the Gegenbauer differential equation [4, p. 783]

$$(1-\eta^2)T_n^{\alpha''}(\eta)-2(\alpha+1)\eta T_n^{\alpha'}(\eta)+n(n+2\alpha+1)T_n^\alpha(\eta)=0. \tag{4}$$

The Gegenbauer functions for $\alpha=m=0, 1, 2, 3, \ldots$ are related to the associated Legendre functions $P_{m+n}^m(\eta)$ by

$$T_n^m(\eta)=\frac{P_{m+n}^m(\eta)}{(1-\eta^2)^{m/2}}, \tag{5}$$

and for $\alpha=-\frac{1}{2}$ and $+\frac{1}{2}$ they are related to the Chebyshev polynomials of the first and second kinds, respectively, by

$$nT_n^{-\frac{1}{2}}(\eta)=\sqrt{\frac{2}{\pi}}\cos\left[n\cos^{-1}\eta\right], \tag{6}$$

$$T_n^{\frac{1}{2}}(\eta)=\sqrt{\frac{2}{\pi}}\,\frac{\sin\left[(n+1)\cos^{-1}\eta\right]}{\sqrt{1-\eta^2}}. \tag{7}$$

(c) For $c\to 0$ and $\eta\to\infty$ such that $c\eta$ remains finite the function $\psi_{\alpha n}(c,\eta)$ becomes proportional to $J_{n+\alpha+\frac{1}{2}}(c\eta)/(c\eta)^{\alpha+\frac{1}{2}}$. This follows from the integral representation (2) in which $\psi_{\alpha n}(c,t)$ in the interval $-1\leq t\leq 1$ becomes proportional to $T_n^\alpha(\eta)$, from the fact [1(20)] that

$$\int_{-1}^{1} e^{ic\eta t}(1-t^2)^\alpha T_n^\alpha(t)\,dt=\frac{\sqrt{2\pi}\,i^n\Gamma(n+2\alpha+1)}{n!}\,\frac{J_{n+\alpha+\frac{1}{2}}(c\eta)}{(c\eta)^{\alpha+\frac{1}{2}}}, \tag{8}$$

for $\alpha>-1$.

(d) The functions $\psi_{\alpha n}(c,\eta)$ are real for real η, have exactly n zeros in the interval $(-1,1)$, and are even or odd functions of η according as n is even or odd. This is implied (not proved) by the limiting case $c\to 0$ in which the limiting functions, the Gegenbauer functions of (4), are known from Sturm-Liouville theory to be real, to have exactly n zeros in $(-1,1)$, and to be even or odd functions of η according as n is even or odd.

(e) The function $\psi_{\alpha n}(c,\eta)$ cannot have a zero at $\eta=\pm 1$. This follows from the differential equation (1) and the entirety of $\psi_{\alpha n}(c,\eta)$. For if it were to have a zero then, from (1), the first derivative also must be zero there. And after l successive differentiations of (1) every $l+1$th derivative must also be zero there, which would require that the function $\psi_{\alpha n}(c, \eta)$ be identically zero everywhere in η.

(f) The eigenvalues $b_{\alpha n}(c)$ of (1) are real, positive, and ordered such that $b_{\alpha 0} < b_{\alpha 1} < b_{\alpha 2} < \ldots$, provided that they are distinct. This is implied (again not proved) by the limiting case $c \to 0$, being given in the limit by (3).

(g) The functions $\psi_{\alpha n}(c, \eta)$ behave asymptotically with real $\eta \to +\infty$ as

$$\psi_{\alpha n}(c, \eta) = \frac{i^n 2^{\alpha+1}\Gamma(\alpha+1)\,\psi_{\alpha n}(c, 1)}{\nu_{\alpha n}(c)} \frac{\cos\left(c\eta - \dfrac{n+\alpha+1}{2}\pi\right)}{(c\eta)^{\alpha+1}}\left[1+O\left(\frac{1}{\eta}\right)\right]. \tag{9}$$

This follows from the known asymptotic behavior [5] of integrals of the form (2).

(h) The spheriodal wave functions $S_{m,\, m+n}(c, \eta)$ and the periodic Mathieu functions $Se_n(c, \eta)$ and $So_{n+1}(c, \eta)$ are special cases related to $\psi_{\alpha n}(c, \eta)$ as follows [2]:

$$\psi_{\alpha n}(c, \eta) = \begin{cases} \dfrac{S_{m,\, m+n}(c, \eta)}{(1-\eta^2)^{m/2}}, & \alpha = m = 0, 1, 2, 3, \ldots \\[2ex] Se_n(c, \eta), & \alpha = -\dfrac{1}{2} \\[2ex] \dfrac{So_{n+1}(c, \eta)}{(1-\eta^2)^{1/2}}, & \alpha = +\dfrac{1}{2}. \end{cases} \tag{10}$$

This is evident from the integral equation for the spheroidal wave functions [6(4)] and for the periodic Mathieu functions [6(11, 12)].

3. Construction of the Functions

On the interval $(-1, 1)$ the Gegenbauer functions $T_k^\alpha(\eta)$ form a complete set with respect to all functions that are square-integrable with weight factor $(1-\eta^2)^\alpha$, and are orthogonal with weight factor $(1-\eta^2)^\alpha$[1(6)]:

$$\int_{-1}^{1} (1-\eta^2)^\alpha T_k^\alpha(\eta) T_l^\alpha(\eta)\, d\eta = \frac{2\Gamma(k+2\alpha+1)}{(2k+2\alpha+1)k!}\delta_{kl}. \tag{11}$$

They are even or odd functions according as k is even or odd. Hence the spheroidal functions $\psi_{\alpha n}(c, \eta)$ can be expanded in Gegenbauer functions on $(-1, 1)$ [1(8)],

$$\psi_{\alpha n}(c, \eta) = \sum_{k=0,1}^{\infty}{}' d_k(c|\alpha n) T_k^\alpha(\eta), \tag{12}$$

where the prime denotes summation over only even or odd values of k according as n is even or odd. The function $\psi_{\alpha n}(c, \eta)$ becomes proportional to $T_n^\alpha(\eta)$ as $c \to 0$, hence

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[2] The connection between the notation of Meixner and Schäfke [3] and that used here is as follows:

$$b_{\alpha n}(c) = \lambda_{\alpha+n}^{\alpha}(c^2) - \alpha(\alpha+1) + c^2, \qquad \psi_{\alpha n}(c, \eta) \propto (1-\eta^2)^{-\alpha/2} ps_{\alpha+n}^{-\alpha}(\eta; c^2);$$

$$b_{-\frac{1}{2}, n}(c) = \lambda_n\left(\frac{c^2}{4}\right) + \frac{c^2}{2}, \qquad \psi_{-\frac{1}{2}, n}(c, \eta) \propto ce_n\left(\cos^{-1}\eta; \frac{c^2}{4}\right);$$

$$b_{\frac{1}{2}, n}(c) = \lambda_{-n-1}\left(\frac{c^2}{4}\right) - 1 + \frac{c^2}{2}, \qquad \psi_{\frac{1}{2}, n}(c, \eta) \propto (1-\eta^2)^{-\frac{1}{2}} se_{n+1}\left(\cos^{-1}\eta; \frac{c^2}{4}\right).$$

$$d_k(c|\alpha n) \to 0,\ k \neq n. \tag{13}$$

Expansion (12) converges rapidly on the interval $(-1, 1)$, but beyond that interval convergence becomes slower and slower with increasing η. An alternative expansion that represents the functions uniformly on $(-\infty, \infty)$, by an argument used later for (40), can be obtained from (12) in the integral equation (2) by integrating term by term and making use of (8),

$$\psi_{\alpha n}(c, \eta) = \frac{i^n\sqrt{2\pi}}{\nu_{\alpha n}(c)} \sum_{k=0,\,1}^{\infty}{}' a_k(c|\alpha n) \frac{J_{k+\alpha+\frac{1}{2}}(c\eta)}{(c\eta)^{\alpha+\frac{1}{2}}}, \tag{14}$$

where the expansion coefficients a_k are related to the d_k by

$$a_k(c|\alpha n) = i^{k-n} \frac{\Gamma(k+2\alpha+1)}{k!} d_k(c|\alpha n). \tag{15}$$

The coefficients a_k are all real because the d_k are real (the factor i^{k-n} is always real since $k-n$ is always even). As $c \to 0$,

$$a_k(c|\alpha n) \to 0,\ k \neq n, \tag{16}$$

by (13). For the special cases of $\alpha = m = 0, 1, 2, 3, \ldots$ the coefficients a_k will agree with those tabulated by Stratton et al. [7] for the prolate spheroidal wave functions provided that the normalization is chosen to be such that

$$\sum_{k=0,\,1}^{\infty}{}' i^{k-n} a_k(c|\alpha n) = 1. \tag{17}$$

With the right hand side of (17) replaced by $\sqrt{\pi/2}$ the coefficients a_k for the special cases of $\alpha = -\frac{1}{2}$ and $+\frac{1}{2}$ are related to Blanch's [8] Mathieu function coefficients $De_k^{(n)}$ and $Do_{k+1}^{(n+1)}$, respectively, by

$$\begin{aligned} a_k(c|-\tfrac{1}{2}, n) &= i^{k-n} \sqrt{\frac{\pi}{2}}\, De_k^{(n)}(c^2), \\ a_k(c|+\tfrac{1}{2}, n) &= i^{k-n} \sqrt{\frac{\pi}{2}}\, (k+1)\, Do_{k+1}^{(n+1)}(c^2). \end{aligned} \tag{18}$$

In addition to the increasingly poor convergence of (12) outside of $(-1, 1)$ it requires special consideration for the case of $\alpha = -\frac{1}{2}$ because $T_k^{-\frac{1}{2}}(\eta)$ is infinite for $k=0$. Hence the expansion (14) will be used almost exclusively hereafter.

A recursion relationship [1(9)] for determining the a_k's in (14) and the eigenvalues $b_{\alpha n}(c)$ is obtained from the differential equation (1),

$$A_{k+2}a_{k+2} + (b_{\alpha n} - B_k)a_k + C_{k-2}a_{k-2} = 0, \tag{19}$$

where

$$A_k=\frac{k(k-1)}{(2k+2\alpha-1)(2k+2\alpha+1)}c^2,$$

$$B_k=k(k+2\alpha+1)+\frac{2k^2+2k(2\alpha+1)+2\alpha-1}{(2k+2\alpha-1)(2k+2\alpha+3)}c^2,$$

$$C_k=\frac{(k+2\alpha+1)(k+2\alpha+2)}{(2k+2\alpha+1)(2k+2\alpha+3)}c^2.$$

It can be rewritten as a continued fraction for the ratio of two succeeding values of a_k in two different ways, one in terms of increasing subscript,

$$\frac{a_k}{a_{k-2}}=-\frac{C_{k-2}}{b_{\alpha n}-B_k+A_{k+2}\dfrac{a_{k+2}}{a_k}},\tag{20}$$

and the other in terms of decreasing subscript,

$$\frac{a_{k-2}}{a_k}=-\frac{A_k}{b_{\alpha n}-B_{k-2}+C_{k-4}\dfrac{a_{k-4}}{a_{k-2}}}.\tag{21}$$

Each must be the reciprocal of the other, which leads to an infinite set of eigenvalues for $b_{\alpha n}$. The best method now available for computing these eigenvalues, and simultaneously the expansion coefficients a_k, is an iterative procedure that was first used by Bouwkamp [9(48)] for the case $\alpha=0$ and later invented independently by Blanch [10] for the Mathieu functions. All of the coefficients a_k in (19) must vanish for negative k, according to a result of Gegenbauer's [11, Sec 16.13] in which he showed that any analytic function can be represented within its domain of analyticity by a series of the form (14). And for $k=0$ or 1 the A_k, B_k, C_k in (19) become indeterminate at $\alpha=\pm\frac{1}{2}$. Therefore (19) should be supplemented by

$$\begin{gathered} a_k=0 \qquad \text{for} \qquad k<0, \\ B_0=\frac{c^2}{2\alpha+3},\ B_1=2\alpha+2+\frac{3c^2}{2\alpha+5},\ C_0=\frac{2\alpha+2}{2\alpha+3}c^2, \end{gathered}\tag{22}$$

the latter of which resolves the indeterminacy. For the special case of the Mathieu functions ($\alpha=\pm\frac{1}{2}$) as computed by Chu and Stratton [2(100)], where $A_k=C_k=\dfrac{c^2}{4}$ and $B_k=k^2+\dfrac{c^2}{2}$, the two computational procedures are quite different although the numerical results have been found to be exactly the same. The advantage of the new procedure here is that all orders $\alpha>-1$ are treated exactly alike.

Once the eigenvalues are known the ratio $\dfrac{a_k}{a_{k-2}}$ for all k is also known, either from (20) or (21). The a_k's can then be computed from these ratios in terms of a_0 or a_1 for the even and odd spheroidal functions, respectively. The particular value chosen for a_0 or a_1 determines the scale factor by which the $\psi_{\alpha n}(c,\eta)$ functions are normalized. But no common agreement exists among authors on the choice of normalization factors. For this reason *all formulas in this paper will be shown with arbitrary normalization factors* that can be chosen at will. For those numerical results shown whose values depend upon a choice of scale factor the $\psi_{\alpha n}(c,\eta)$ functions will be normalized to unity over the interval $(-\infty,\infty)$.

The eigenvalues $\nu_{\alpha n}(c)$ can be computed by equating the two expressions (12) and (14), or their derivatives, at $\eta=0$. As η approaches zero the Bessel functions in (14) cause all terms to vanish except for $k=0$. Hence for even $n=2r$,

$$\nu_{\alpha,2r}(c)=\frac{(-1)^r\sqrt{\pi}\,a_0(c|\alpha,2r)}{2^\alpha\Gamma(\alpha+\frac{3}{2})\psi_{\alpha,2r}(c,0)}. \tag{23}$$

For odd $n=2r+1$ the $\psi_{\alpha n}(c,\eta)$ vanish at $\eta=0$, so (14) must be differentiated with respect to η before letting η approach zero. All terms vanish except for $k=1$, hence

$$\nu_{\alpha,2r+1}(c)=ic\,\frac{(-1)^r\sqrt{\pi}\,a_1(c|\alpha,2r+1)}{2^{\alpha+1}\Gamma(\alpha+\frac{5}{2})\psi'_{\alpha,2r+1}(c,0)}. \tag{24}$$

The $\psi_{\alpha n}(c,\eta)$ functions and their derivatives at $\eta=0$ are found from (12) to be

$$\psi_{\alpha,2r}(c,0)=\frac{(-1)^r}{2^\alpha\sqrt{\pi}}\sum_{l=0}^{\infty}\frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+\alpha+1)}\,a_{2l}(c|\alpha,2r), \tag{25}$$

$$\psi'_{\alpha,2r+1}(c,0)=\frac{(-1)^r}{2^{\alpha-1}\sqrt{\pi}}\sum_{l=0}^{\infty}\frac{\Gamma(l+\frac{3}{2})}{\Gamma(l+\alpha+1)}\,a_{2l+1}(c|\alpha,2r+1). \tag{26}$$

Also,

$$\nu_{\alpha n}(c)=i^n|\nu_{\alpha n}(c)|. \tag{27}$$

4. Double Orthogonality

An important property that will be established here is that the spheroidal functions $\psi_{\alpha n}(c,\eta)$ for any order $\alpha>-1$ are orthogonal with weight factor $|1-\eta^2|^\alpha$ on two different intervals $(-1,1)$ and $(-\infty,\infty)$ simultaneously. This double orthogonality property was first recognized for the case of the prolate spheroidal wave functions of order zero by Slepian and Pollak [12]. A later investigation by the author [6] revealed that this property was possessed by all of the prolate spheroidal wave functions of order $m=0,1,2,3,\ldots$ and by the periodic Mathieu functions.

The differential equation for the functions $S_{\alpha,\alpha+n}(c,\eta)$ defined by

$$\psi_{\alpha n}(c,\eta)=\frac{S_{\alpha,\alpha+n}(c,\eta)}{(1-\eta^2)^{\alpha/2}} \tag{28}$$

is found from (1) to be of the singular Sturm-Liouville form

$$[(1-\eta^2)S'_{\alpha,\alpha+n}(c,\eta)]'+\left[b_{\alpha n}+\alpha(\alpha+1)-c^2\eta^2-\frac{\alpha^2}{1-\eta^2}\right]S_{\alpha,\alpha+n}(c,\eta)=0. \tag{29}$$

Multiplication by $S^*_{\alpha,\alpha+p}(c,\eta)$ and subtraction of the product equation obtained when $S_{\alpha,\alpha+n}(c,\eta)$ and $S^*_{\alpha,\alpha+p}(c,\eta)$ are interchanged gives

$$\frac{d}{d\eta}\left[(1-\eta^2)(S'_{\alpha,\alpha+n}S^*_{\alpha,\alpha+p}-S_{\alpha,\alpha+n}S^{*\prime}_{\alpha,\alpha+p})\right]=(b_{\alpha p}-b_{\alpha n})S_{\alpha,\alpha+n}S^*_{\alpha,\alpha+p}. \tag{30}$$

Integrating with respect to η over a real interval and changing from $S_{\alpha,\alpha+n}(c,\eta)$ back to $\psi_{\alpha n}(c,\eta)$,

$$(1-\eta^2)|1-\eta^2|^\alpha[\psi'_{\alpha n}(c,\eta)\psi_{\alpha p}(c,\eta)-\psi_{\alpha n}(c,\eta)\psi'_{\alpha p}(c,\eta)]\Big|_{\eta=\eta_1}^{\eta_2}$$
$$=(b_{\alpha p}-b_{\alpha n})\int_{\eta_1}^{\eta_2}\psi_{\alpha n}(c,\eta)\psi_{\alpha p}(c,\eta)|1-\eta^2|^\alpha d\eta. \tag{31}$$

The functions $\psi_{\alpha n}(c,\eta)$ will be orthogonal over any real interval (η_1,η_2) for which the left hand side of (31) is zero, provided that the $b_{\alpha n}$ are distinct. Distinctness of the $b_{\alpha n}$ can usually be determined numerically.

(a) *Orthogonality on* $(-1, 1)$

When $\eta_1=-1$ and $\eta_2=1$ the left hand side of (31) vanishes for all $\alpha>-1$ because $(1-\eta^2)|1-\eta^2|^\alpha$ vanishes at both end points and $\psi_{\alpha n}(c, \eta)$ and its derivative are finite. Hence, if the $b_{\alpha n}$ are distinct,

$$\int_{-1}^{1} \psi_{\alpha n}(c, \eta)\psi_{\alpha p}(c, \eta)(1-\eta^2)^\alpha d\eta=0, \qquad n\neq p; \tag{32}$$

i.e., the $\psi_{\alpha n}(c, \eta)$ functions are orthogonal on $(-1, 1)$ with weight factor $(1-\eta^2)^\alpha$.

(b) *Orthogonality on* $(-\infty, \infty)$

From the asymptotic behavior (9) of $\psi_{\alpha n}(c, \eta)$ a simple calculation shows that

$$\begin{aligned}(1-\eta^2)|1-\eta^2|^\alpha[\psi'_{\alpha n}(c, \eta)\psi_{\alpha p}(c, \eta)-\psi_{\alpha n}(c, \eta)\psi'_{\alpha p}(c, \eta)]\\ =\frac{\psi_{\alpha n}(c, 1)\psi_{\alpha p}(c, 1)4^{\alpha+1}\Gamma^2(\alpha+1)}{|\nu_{\alpha n}(c)||\nu_{\alpha p}(c)|c^{2\alpha+1}}\left[\sin\frac{p-n}{2}\pi+\mathrm{O}\left(\frac{1}{\eta}\right)\right],\end{aligned} \tag{33}$$

as $\eta\rightarrow+\infty$, η real. When $p-n$ is even the first term is zero for all η, leaving

$$(1-\eta^2)|1-\eta^2|^\alpha[\psi'_{\alpha n}(c, \eta)\psi_{\alpha p}(c, \eta)-\psi_{\alpha n}(c, \eta)\psi'_{\alpha p}(c, \eta)]=\mathrm{O}(1/\eta).$$

The left hand side of (31) then vanishes for $\eta_1=1$ and $\eta_2=\infty$, hence

$$\int_{1}^{\infty} \psi_{\alpha n}(c, \eta)\psi_{\alpha p}(c, \eta)|1-\eta^2|^\alpha d\eta=0, \qquad p-n \text{ even and} \neq 0. \tag{34}$$

Similarly on the interval $(-\infty, -1)$, because of symmetry of the functions. Combining the three intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$,

$$\int_{-\infty}^{\infty} \psi_{\alpha n}(c, \eta)\psi_{\alpha p}(c, \eta)|1-\eta^2|^\alpha d\eta=0, \qquad p\neq n, \tag{35}$$

including $p-n$ odd because $\psi_{\alpha n}(c, \eta)\psi_{\alpha p}(c, \eta)$ is then an odd function on $(-\infty, \infty)$. Thus the functions $\psi_{\alpha n}(c, \eta)$ are orthogonal with weight factor $|1-\eta^2|^\alpha$ on the interval $(-\infty, \infty)$, provided only that the $b_{\alpha n}$ are distinct.

5. Completeness

(a) *Completeness on* $(-1, 1)$

The functions $S_{\alpha, \alpha+n}(c, \eta)$ are complete on the interval $(-1, 1)$ with respect to all square-integrable functions. This is a property [13] of all Sturm-Liouville systems of the form

$$(pu')' + (\lambda w - q)u = 0, \quad puu'|_a = puu'|_b = 0, \tag{36}$$

and hence applies to the differential equation (29) and the boundary condition that $S_{\alpha,\alpha+n}(c, \eta)$ must remain finite at the singular points $\eta = \pm 1$. Completeness of the $S_{\alpha,\alpha+n}(c, \eta)$ functions implies completeness of the $\psi_{\alpha n}(c, \eta)$ functions.

(b) *Completeness on* $(-\infty, \infty)$

The fact that the functions $\psi_{\alpha n}(c, \eta)$ are complete on $(-1, 1)$ with respect to the class of functions that are square-integrable with weight factor $(1-\eta^2)^\alpha$ will now be used to show that they are also complete on $(-\infty, \infty)$ with respect to all functions of the form

$$F_\alpha(c\eta) = \int_{-1}^{1} e^{ic\eta t}(1-t^2)^\alpha f(t)\,dt \tag{37}$$

for $\alpha > -1$. The only restriction on $f(t)$ is that it be square-integrable on $(-1, 1)$ with weight factor $(1-t^2)^\alpha$. It can be expanded in a series that converges in the mean with weight factor $(1-t^2)^\alpha$,

$$f(t) = \sum_{n=0}^{\infty} \frac{b_n(c)}{\nu_{\alpha n}(c)} \psi_{\alpha n}(c, t), \qquad -1 < t < 1, \tag{38}$$

where

$$b_n(c) = \frac{\nu_{\alpha n}(c)}{\Lambda_{\alpha n}(c)} \int_{-1}^{1} f(t)\psi_{\alpha n}(c, t)(1-t^2)^\alpha dt. \tag{39}$$

Hence for *all* real values of η the absolute value

$$\begin{aligned}\left|F_\alpha(c\eta) - \sum_{n=0}^{\infty} b_n\psi_{\alpha n}(c, \eta)\right| &= \left|\int_{-1}^{1} e^{ic\eta t}(1-t^2)^\alpha \left[f(t) - \sum_{n=0}^{\infty} \frac{b_n}{\nu_{\alpha n}} \psi_{\alpha n}(c, t)\right] dt\right| \\ &\leqslant \int_{-1}^{1} (1-t^2)^\alpha \left|f(t) - \sum_{n=0}^{\infty} \frac{b_n}{\nu_{\alpha n}} \psi_{\alpha n}(c, t)\right| dt\end{aligned} \tag{40}$$

is exactly zero. It is concluded, therefore, that

$$F_\alpha(c\eta) = \sum_{n=0}^{\infty} b_n(c)\psi_{\alpha n}(c, \eta), \qquad -\infty < \eta < \infty, \tag{41}$$

where the series converges to $F_\alpha(c\eta)$ not just in the mean but uniformly on $(-\infty, \infty)$. Thus the functions $\psi_{\alpha n}(c, \eta)$ are complete on $(-\infty, \infty)$ with respect to all functions $F_\alpha(c\eta)$ of the form (37).

6. Normalization Factors

The double orthogonality property of the spheroidal functions results in two different sets of normalization factors, one on $(-1, 1)$ and another on $(-\infty, \infty)$. Both will be developed here.

(a) *Normalization factors on* $(-1, 1)$.

The orthonormality relation on $(-1, 1)$ will be designated by

$$\int_{-1}^{1} \psi_{\alpha n}(c, \eta)\psi_{\alpha p}(c, \eta)(1-\eta^2)^\alpha d\eta = \Lambda_{\alpha n}(c)\delta_{np}, \tag{42}$$

where $\Lambda_{\alpha n}(c)$ denotes the normalization factor for the spheroidal function of order α and index n. These normalization factors can be computed directly from the integral (42) for each $p = n$ by expanding $\psi_{\alpha n}(c,\eta)$ in Gegenbauer functions as in (12) and making use of their orthogonality relation (11), obtaining

$$\Lambda_{\alpha n}(c)=\sum_{k=0,1}^{\infty}{}' \frac{k!}{(k+\alpha+\frac{1}{2})\Gamma(k+2\alpha+1)} a_k^2(c|\alpha n), \tag{43}$$

where the a_k's are the expansion coefficients (15).

(b) *Normalization factors on* $(-\infty,\infty)$.

Each of the normalization factors on $(-\infty,\infty)$ can be derived in terms of the corresponding normalization factor $\Lambda_{\alpha n}(c)$ on $(-1,1)$ by making use of the double orthogonality property already established. The orthogonality integral (35) can be written

$$\int_{-\infty}^{\infty} \psi_{\alpha n}(c,\eta)\psi_{\alpha p}(c,\eta)|1-\eta^2|^{\alpha}d\eta$$
$$=\int_{-1}^{1}\left[\frac{1}{\nu_{\alpha p}(c)}\int_{-\infty}^{\infty} e^{ic\eta t}|1-\eta^2|^{\alpha}\psi_{\alpha n}(c,\eta)d\eta\right](1-t^2)^{\alpha}\psi_{\alpha p}(c,t)dt, \tag{44}$$

by replacing $\psi_{\alpha p}(c,\eta)$ by its integral representation (2) and interchanging the order of integration. From the orthogonality property (35) on $(-\infty,\infty)$ it is clear that this must be zero for all $p\neq n$. And from the other orthogonality property (32) on $(-1,1)$ it follows that the bracketed expression in (44) must be orthogonal to every $\psi_{\alpha p}(c,t)$ except $\psi_{\alpha n}(c,t)$. Then, from completeness on $(-1,1)$, it is concluded that the bracketed expression for $p=n$ must be proportional to $\psi_{\alpha n}(c,t)$ everywhere on $(-1,1)$:

$$\frac{1}{\nu_{\alpha n}(c)}\int_{-\infty}^{\infty} e^{ic\eta t}|1-\eta^2|^{\alpha}\psi_{\alpha n}(c,\eta)d\eta\equiv\gamma_{\alpha n}(c)\psi_{\alpha n}(c,t),\qquad -1<t<1, \tag{45}$$

where $\gamma_{\alpha n}(c)$ is the proportionality constant.[3] Thus the orthogonality relation on $(-\infty,\infty)$ is

$$\int_{-\infty}^{\infty}\psi_{\alpha n}(c,\eta)\psi_{\alpha p}(c,\eta)|1-\eta^2|^{\alpha}d\eta=\gamma_{\alpha n}(c)\Lambda_{\alpha n}(c)\delta_{np}; \tag{46}$$

that is, each normalization factor on $(-\infty,\infty)$ is proportional to the corresponding normalization factor on $(-1,1)$, the proportionality factor being the real positive number $\gamma_{\alpha n}(c)$ defined by (45).

The new numbers $\gamma_{\alpha n}(c)$ representing the ratio of the two normalization factors are characteristic of the functions $\psi_{\alpha n}(c,\eta)$, much like the eigenvalues $b_{\alpha n}(c)$ and $\nu_{\alpha n}(c)$, even though their defining equation (45) is not an integral equation in any ordinary sense. However they are by far the most difficult of the three to evaluate numerically for arbitrary $\alpha>-1$. In a few special cases, notably $\alpha=-\frac{1}{2}$, 0, $\frac{1}{2}$, and 1, it becomes practical to evaluate them by evaluating the normalization integral (46). Such a method was actually used to obtain the first published values for the case of the prolate spheroidal wave functions of order zero [12, p. 59] (they were not recognized there as being a new kind of characteristic numbers because for $\alpha=0$ they are not, being related simply to the eigenvalues of integral equation (2)) and the Mathieu functions [6 (60, 61)]. But for arbitrary α the only practical method to evaluate them appears to be from their definition (45),

$$\gamma_{\alpha n}(c)=\frac{\int_{-\infty}^{\infty} e^{ic\eta t}|1-\eta^2|^{\alpha}\psi_{\alpha n}(c,\eta)d\eta}{\nu_{\alpha n}(c)\psi_{\alpha n}(c,t)},\qquad -1<t<1, \tag{47}$$

[3] An early draft of the present paper was communicated privately to J. Meixner, who became sufficiently concerned about the lack of convergence of the integral in (45) for $\alpha\geqslant 1$ that he looked into the possibility of an alternative derivation. From some unpublished notes that he kindly sent me after this paper was submitted for publication it appears that he has succeeded in deriving the relationship (45) by an entirely different approach. He has given the integral from $-\infty$ to -1 and from 1 to ∞ a meaning by assuming that t has a negative or positive imaginary part, respectively, and then approaching the limit of real t, thereby avoiding the difficult problem of proving summability.

at some convenient value of t in the interval $(-1, 1)$. In particular, at $t=0$ this reduces to

$$\gamma_{\alpha,2r}(c)-1=\frac{2\int_1^\infty (\eta^2-1)^\alpha \psi_{\alpha,2r}(c,\eta)\,d\eta}{\nu_{\alpha,2r}(c)\psi_{\alpha,2r}(c,0)} \tag{48}$$

for even $n=2r$. For odd $n=2r+1$ the ratio (47) becomes indeterminate at $t=0$ because $\psi_{\alpha,2r+1}(c,\eta)$ is an odd function of η. It can be made determinate by applying L'Hospital's rule at $t=0$,

$$\gamma_{\alpha,2r+1}(c)-1=\frac{i2c\int_1^\infty \eta(\eta^2-1)^\alpha \psi_{\alpha,2r+1}(c,\eta)\,d\eta}{\nu_{\alpha,2r+1}(c)\psi'_{\alpha,2r+1}(c,0)} \tag{49}$$

for $n=2r+1$. The problem of evaluating $\gamma_{\alpha n}(c)$ thus reduces to evaluation of the integrals in (48) and (49).

The above method for determining $\gamma_{\alpha n}(c)$ was introduced originally by the author [6(42)] for the special case of the spheroidal wave functions of integral order $\alpha=m$. It has been confirmed and generalized recently in an elegant and quite different way by Meixner [14], who showed that for integral $\alpha=m$ an expression equivalent to (45) holds for the most general solutions of the differential equation (1) for arbitrary complex values of c, of b, and of the characteristic exponent ν in place of the integer n.

The integral in (48) does not converge at infinity for $\alpha \geqslant 1$, nor does the one in (49) for $\alpha \geqslant 0$, but they both appear to be Cesàro summable $(C, m+1)$ for all $-1<\alpha \leqq m$ because of the sinusoidal asymptotic behavior (9) of $\psi_{\alpha n}(c,\eta)$. They can be evaluated by expanding $\psi_{\alpha n}(c,\eta)$ in Bessel functions by (14) and integrating term by term,

$$\gamma_{\alpha,2r}(c)-1=\frac{2\sqrt{2\pi}}{|\nu^2_{\alpha,2r}(c)\psi_{\alpha,2r}(c,0)|}\sum_{p=0}^{\infty} a_{2p}(c|\alpha,2r)I_{\alpha,2p}(c), \tag{50}$$

$$\gamma_{\alpha,2r+1}(c)-1=\frac{2\sqrt{2\pi}}{|\nu^2_{\alpha,2r+1}(c)\psi'_{\alpha,2r+1}(c,0)|}\sum_{p=0}^{\infty} a_{2p+1}(c|\alpha,2r+1)K_{\alpha,2p}(c), \tag{51}$$

where the integrals $I_{\alpha k}(c)$ and $K_{\alpha k}(c)$ are defined by

$$I_{\alpha k}(c)=\int_1^\infty (\eta^2-1)^\alpha \frac{J_{k+\alpha+\frac{1}{2}}(c\eta)}{(c\eta)^{\alpha+\frac{1}{2}}}\,d\eta, \tag{52}$$

$$K_{\alpha k}(c)=\int_1^\infty (\eta^2-1)^\alpha \frac{J_{k+2+\alpha-\frac{1}{2}}(c\eta)}{(c\eta)^{\alpha-\frac{1}{2}}}\,d\eta. \tag{53}$$

For $k=0$ the integral $I_{\alpha 0}(c)$ in the range $-1<\alpha<0$ can be evaluated by replacing the Bessel function in (52) by its representation [11, p. 48(2)] as an integral on $(0, 1)$ and interchanging the order of integration,

$$I_{\alpha 0}(c)=\frac{2^{-\alpha+\frac{1}{2}}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2})}\int_0^1 (1-t^2)^\alpha \left[\int_1^\infty (\eta^2-1)^\alpha \cos c\eta t\,d\eta\right]dt. \tag{54}$$

The inner integral on $(1,\infty)$ is an integral representation for the Neumann function [11, p. 170(4)], hence

$$I_{\alpha 0}(c) = -\int_0^1 (1-t^2)^\alpha \frac{Y_{-(\alpha+\frac{1}{2})}(ct)}{(ct)^{\alpha+\frac{1}{2}}}\,dt = -\frac{1}{\cos\alpha\pi}\left[\sin\alpha\pi\int_0^1 (1-t^2)^\alpha \frac{J_{-(\alpha+\frac{1}{2})}(ct)}{(ct)^{\alpha+\frac{1}{2}}}\,dt + \int_0^1 (1-t^2)^\alpha \frac{J_{\alpha+\frac{1}{2}}(ct)}{(ct)^{\alpha+\frac{1}{2}}}\,dt\right], \tag{55}$$

the latter expression coming from the definition of the Neumann function [11, p. 64(1)]. Expanding the Bessel functions in power series about the origin and integrating term by term gives

$$I_{\alpha 0}(c) = \frac{\Gamma(\alpha+1)}{2^{\alpha+\frac{3}{2}}\cos\alpha\pi}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}\left[\frac{\sin\alpha\pi\Gamma(l-\alpha)}{l!\Gamma(l-\alpha+\frac{1}{2})}\left(\frac{c}{2}\right)^{-2\alpha-1} + \frac{\Gamma(l+\frac{1}{2})}{\Gamma^2(l+\alpha+\frac{3}{2})}\right]\left(\frac{c}{2}\right)^{2l}. \tag{56}$$

The integrals $I_{\alpha k}(c)$ and $K_{\alpha k}(c)$ for all k can now be obtained from $I_{\alpha 0}(c)$ by recursion in k. By use of the recursion relationships among Bessel functions [11, p. 45(1,2)] the following recursion relationships among $I_{\alpha k}(c)$ and $K_{\alpha k}(c)$ can be established, valid for all $\alpha > -1$:

$$K_{\alpha,k-2}(c) + K_{\alpha k}(c) = (2k+2\alpha+1) I_{\alpha k}(c), \tag{57}$$

$$K_{\alpha,k-2}(c) - K_{\alpha k}(c) = (2\alpha+1) I_{\alpha k}(c) + 2cI'_{\alpha k}(c), \tag{58}$$

$$K_{\alpha k}(c) = kI_{\alpha k}(c) - cI'_{\alpha k}(c), \tag{59}$$

$$(k+2\alpha+3) I_{\alpha,k+2}(c) + cI'_{\alpha,k+2}(c) = kI_{\alpha k}(c) - cI'_{\alpha k}(c). \tag{60}$$

The last relation (60) is a differential equation for $I_{\alpha,k+2}(c)$ in terms of $I_{\alpha k}(c)$. Its solution is

$$I_{\alpha,k+2}(c) = \frac{2k+2\alpha+3}{c^{k+2\alpha+3}}\int_0^c t^{k+2\alpha+2} I_{\alpha k}(t)\,dt - I_{\alpha k}(c). \tag{61}$$

Repeated application of (61) to $I_{\alpha 0}(c)$ in (56) suggests the appropriate form of the power series expansion for $I_{\alpha,2p}(c)$, which is then easily shown by induction to hold for all nonnegative integers p. Its jth derivative, obtained by differentiating term by term with respect to c, is

$$I^{(j)}_{\alpha,2p}(c) = \frac{\Gamma(\alpha+1)}{2^{j+\alpha+\frac{3}{2}}\cos\alpha\pi}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}\left[\frac{\sin\alpha\pi\Gamma(l-\alpha)(-l+\alpha+\frac{1}{2})_p\,(2l-2\alpha-j)_j}{(l+p)!\Gamma(l-\alpha+\frac{1}{2})}\left(\frac{c}{2}\right)^{-2\alpha-1} + \frac{\Gamma(l+\frac{1}{2})(-l)_p(2l+1-j)_j}{\Gamma(l+\alpha+\frac{3}{2})\Gamma(l+\alpha+\frac{3}{2}+p)}\right]\left(\frac{c}{2}\right)^{2l-j}, \tag{62}$$

where $(z)_p = z(z+1)(z+2)\ \ldots\ (z+p-1)$, $(z)_0 = 1$, is Pochhammer's symbol. $I_{\alpha,2p}(c)$ and $K_{\alpha,2p}(c)$ for $-1 < \alpha < 0$ can thus be computed by (62) and (59), respectively.

To obtain $I_{\alpha,2p}(c)$ and $K_{\alpha,2p}(c)$ for orders beyond the region of integrability $-1 < \alpha < 0$ of (55) the following recursion relations are valid for all $\alpha > 0$,

$$I_{\alpha k}(c) = \left(\frac{k}{c^2} - \frac{1}{2k+2\alpha+1}\right) I_{\alpha-1,k}(c) - \frac{1}{2k+2\alpha+1} I_{\alpha-1,k+2}(c) - \frac{1}{c} I'_{\alpha-1,k}(c), \tag{63}$$

$$K_{\alpha k}(c) = \frac{k(k+2)}{c^2} I_{\alpha-1,k}(c) - I_{\alpha-1,k+2}(c) - \frac{2k+1}{c} I'_{\alpha-1,k}(c) + I''_{\alpha-1,k}(c), \tag{64}$$

the first being obtained from the definition (52) and the Bessel function recursion relations and the second from (59). It has been found by direct substitution that the series (62) satisfies the recursion

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relation (63). From this it is concluded that (62) is valid for *all* $\alpha > -1$, even though the integral (55) from which it was derived contains a nonintegrable singularity at $t=0$ for all nonintegral values of α greater than zero. As an indication of the rapidity of convergence of (62) and of (50) and (51), and hence of the practicality of the formulas developed here, the series (62) always was found to converge to 16 significant figures in less than 23 terms for the computations shown later, and the series (50) and (51) to converge to 16 figures in less than 19 terms.

The series (62) becomes indeterminate when α is an integer or a half-integer. The indeterminancy at any of the integers $\alpha=m$ can be resolved by taking the limit as α approaches m. For $\alpha=0$, for example, the limiting value $I_{00}(c)$ is

$$I_{00}(c)=\frac{1}{c}\sqrt{\frac{\pi}{2}}-\sqrt{\frac{2}{\pi}}\sum_{l=0}^{\infty}(-1)^l\frac{c^{2l}}{(2l+1)(2l+1)!}, \tag{65}$$

in agreement with the power series expansion of

$$\frac{1}{c}\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}-Si\ c\right)$$

obtained by direct evaluation of the integral (52). At $\alpha=-\frac{1}{2}$ the indeterminancy again can be resolved by taking the limit, obtaining

$$I_{-\frac{1}{2},0}(c)=\frac{1}{2\sqrt{\pi}}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(l+\frac{1}{2})}{(l!)^3}\left[3\psi(l+1)-\psi(l+\tfrac{1}{2})-2\ln\frac{c}{2}\right]\left(\frac{c}{2}\right)^{2l} \tag{66}$$

in terms of the Digamma function $\psi(z)$, which is identical to the power series expansion [11, p. 150 (1)] of

$$-\frac{\pi}{2}J_0\left(\frac{c}{2}\right)Y_0\left(\frac{c}{2}\right)$$

obtained by direct evaluation of the integral (52). But in the special cases of integral and half-integral orders, and these cases only, there are more direct means available for evaluating $\gamma_{\alpha n}(c)$ than by the series (50) and (51). For all integral orders the numbers $\gamma_{\alpha n}(c)$ are found to be expressible quite simply in terms of the spheroidal eigenvalues $b_{\alpha n}(c)$ of the differential equation and $\nu_{\alpha n}(c)$ of the integral equation [6(23)]. For the half-integral orders $\alpha=\pm\frac{1}{2}$ the numbers $\gamma_{\alpha n}(c)$ have been found [6(31)] in terms of Blanch's joining factors $f_{e,n}(c^2)$ and $f_{o,n+1}(c^2)$ to be simply

$$\gamma_{\alpha n}(c)=\begin{cases}1+f_{e,n}(c^2), & \alpha=-\frac{1}{2}\\ 1+f_{o,n+1}(c^2), & \alpha=+\frac{1}{2}.\end{cases} \tag{67}$$

From Blanch's expansion of $f_{e,n}(c^2)$ and $f_{o,n+1}(c^2)$ in terms of Bessel functions it can be shown that (67) is identical to that obtained from (50) and (51). The integrals $I_{\alpha k}(c)$ and $K_{\alpha k}(c)$ can be expressed in closed form for half-integral values of α, starting from $I_{-\frac{1}{2},2p}(c)$ [15(6.552.6)]

$$I_{-\frac{1}{2},2p}(c)=-\frac{\pi}{2}J_p\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right), \tag{68}$$

from which $K_{-\frac{1}{2},2p}(c)$ is obtained by recursion relation (59),

$$K_{-\frac{1}{2},2p}(c)=-\frac{\pi c}{4}\left[J_p\left(\frac{c}{2}\right)Y_{p+1}\left(\frac{c}{2}\right)+J_{p+1}\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right)\right], \tag{69}$$

and from which $I_{\frac{1}{2},\,2p}(c)$ and $K_{\frac{1}{2},\,2p}(c)$ are obtained by recursion relations (63) and (59), respectively,

$$I_{\frac{1}{2},2p}(c)=\frac{\pi}{4(2p+1)}\left[J_p\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right)+J_{p+1}\left(\frac{c}{2}\right)Y_{p+1}\left(\frac{c}{2}\right)\right.$$
$$\left.-\frac{2p+1}{c}\left(J_p\left(\frac{c}{2}\right)Y_{p+1}\left(\frac{c}{2}\right)+J_{p+1}\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right)\right)\right], \tag{70}$$

$$K_{\frac{1}{2},2p}(c)=\frac{\pi}{4}\left[J_p\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right)+J_{p+1}\left(\frac{c}{2}\right)Y_{p+1}\left(\frac{c}{2}\right)\right.$$
$$\left.-\frac{2p+2}{c}\left(J_p\left(\frac{c}{2}\right)Y_{p+1}\left(\frac{c}{2}\right)+J_{p+1}\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right)\right)\right]. \tag{71}$$

7. An Extremal Property

An important property of the prolate spheroidal wave functions of order zero was established by Landau and Pollak [16, Th. 1], namely, that the first $N+1$ of them are the $N+1$ linearly independent functions of the form $F_0(c\eta)$ that are most concentrated in the interval $(-1, 1)$. By the term "concentration" of a function of the form $F_\alpha(c\eta)$ defined by (37) will be meant the value of the ratio $R_\alpha(c)$,

$$R_\alpha(c)=\frac{\int_{-1}^{1}|F_\alpha(c\eta)|^2(1-\eta^2)^\alpha d\eta}{\int_{-\infty}^{\infty}|F_\alpha(c\eta)|^2|1-\eta^2|^\alpha d\eta}. \tag{72}$$

It will now be shown, by quite a different and much more simple argument, that for *any* order $\alpha>-1$ the first $N+1$ of the spheroidal functions $\psi_{\alpha n}(c,\eta)$ are the $N+1$ linearly independent functions of the form $F_\alpha(c\eta)$ that are most concentrated in the interval $(-1, 1)$.

After expanding $F_\alpha(c\eta)$ in spheroidal functions as in (41) and making use of the two orthogonality relations (42) and (46) the concentration ratio (72) becomes

$$R_\alpha=\frac{\sum_{n=0}^{\infty} b_n b_n^* \Lambda_{\alpha n}}{\sum_{n=0}^{\infty} b_n b_n^* \gamma_{\alpha n}\bar{\Lambda}_{\alpha n}}. \tag{73}$$

It is determined entirely by the expansion coefficients b_n of the function $F_\alpha(c\eta)$. Hence a necessary condition for it to attain its largest value is that it be an extremum with respect to the real and the imaginary parts of each and every one of the b_n's; i.e., since b_n and b_n^* are linearly related to the real and imaginary parts of b_n,

$$\frac{\partial R_\alpha}{\partial b_p}=\frac{\partial R_\alpha}{\partial b_p^*}=0, \qquad p=0, 1, 2, 3, \ldots. \tag{74}$$

Rewriting (73) in the form

$$\sum_{n=0}^{\infty} b_n b_n^*(R_\alpha\gamma_{\alpha n}-1)\Lambda_{\alpha n}=0, \tag{75}$$

then differentiating through with respect to b_p or b_p^* and imposing condition (74),

$$b_p^*(R_\alpha\gamma_{\alpha p}-1)\Lambda_{\alpha p}=b_p(R_\alpha\gamma_{\alpha p}-1)\Lambda_{\alpha p}=0, \qquad p=0, 1, 2, 3, \ldots. \tag{76}$$

The normalization factors $\Lambda_{\alpha p}(c)$ are never zero, hence either

$$b_p = b_p^* = 0, \tag{77}$$

or

$$R_\alpha \gamma_{\alpha p} - 1 = 0. \tag{78}$$

The second condition (78) can be true for at most only one value of p, in which case condition (74) for all other values of p requires that (77) be true. It is clear, then, that an extremum of $R_\alpha(c)$ can occur when, and only when, $F_\alpha(c\eta)$ is proportional to one of the spheroidal functions $\psi_{\alpha n}(c, \eta)$. The value of $R_\alpha(c)$ at each of these extrema is, from (78),

$$R_{\alpha_{\text{ext}}}(c) = \frac{1}{\gamma_{\alpha n}(c)}, \qquad n = 0, 1, 2, 3, \ldots. \tag{79}$$

The largest possible value of $R_\alpha(c)$ is that for which $\gamma_{\alpha n}(c)$ is least, which is $\gamma_{\alpha 0}(c)$. Hence the spheroidal function for $n = 0$ is the single most concentrated function of the form $F_\alpha(c\eta)$.

By the term "linear independence" of a set of functions is meant that they are mutually orthogonal. Thus to find the second most concentrated function of the form $F_\alpha(c\eta)$ that is linearly independent of $\psi_{\alpha 0}(c, \eta)$ on $(-\infty, \infty)$ means to find the most concentrated function of the form $F_\alpha(c\eta)$ that is orthogonal to $\psi_{\alpha 0}(c, \eta)$ on $(-\infty, \infty)$. From the orthogonality of the functions $\psi_{\alpha n}(c, \eta)$ on $(-\infty, \infty)$ and from their completeness with respect to functions of the form $F_\alpha(c\eta)$ it follows that all functions of the form $F_\alpha(c\eta)$ that are linearly independent of $\psi_{\alpha 0}(c, \eta)$ can be expanded in the set of $\psi_{\alpha n}(c, \eta)$ functions that remain after $\psi_{\alpha 0}(c, \eta)$ has been removed; i.e., after setting $b_0 = 0$ in (41). Repeating the maximization process above leads again to conditions (77) and (78) but restricted now to $p = 1, 2, 3 \ldots.$ The largest possible value for $R_\alpha(c)$ is again that for which $\gamma_{\alpha n}(c)$ is least, which now is $\gamma_{\alpha 1}(c)$. Hence the spheroidal functions for $n = 0$ and 1 are the most concentrated *pair* of linearly independent functions of the form $F_\alpha(c\eta)$. By this process of systematically depleting the set of functions $\psi_{\alpha n}(c, \eta)$ it is concluded that the first $N+1$ of them are indeed the $N+1$ linearly independent functions of the form $F_\alpha(c\eta)$ that are most concentrated in the interval $(-1, 1)$.

8. Spheroidal Identities

All of the identities to be derived here will be obtained from the integral equation (2) and the following property of completeness of the functions $\psi_{\alpha n}(c, \eta)$ on $(-1, 1)$:

$$\delta(t - t') = (1 - t^2)^\alpha \sum_{n=0}^{\infty} \frac{\psi_{\alpha n}(c, t)\psi_{\alpha n}(c, t')}{\Lambda_{\alpha n}(c)}, \tag{80}$$

for $-1 < t < 1$ and $-1 < t' < 1$, obtained by expanding the delta function $\delta(t - t')$ in the orthonormal functions $\psi_{\alpha n}(c, t')/\sqrt{\Lambda_{\alpha n}(c)}$ with expansion coefficients $(1 - t^2)^\alpha \psi_{\alpha n}(c, t)/\sqrt{\Lambda_{\alpha n}(c)}$. Although such an expansion is open to question on grounds that the delta function is not square-integrable it does appear to be valid as the limiting case of some square-integrable function that approaches the delta function.

The principal identity from which the others will be derived is obtained by multiplying the integral equation (2) for argument η by its complex conjugate for some other argument y, then dividing by the normalization factor $\Lambda_{\alpha n}(c)$, and finally summing over all n,

$$\sum_{n=0}^{\infty} \frac{|\nu_{\alpha n}(c)|^2}{\Lambda_{\alpha n}(c)} \psi_{\alpha n}(c, \eta)\psi_{\alpha n}(c, y)$$

$$= \int_{-1}^{1} \int_{-1}^{1} e^{ic(\eta t - y t')}(1 - t'^2)^\alpha \left[(1 - t^2)^\alpha \sum_{n=0}^{\infty} \frac{\psi_{\alpha n}(c, t)\psi_{\alpha n}(c, t')}{\Lambda_{\alpha n}(c)} \right] dt dt'. \tag{81}$$

The bracketed expression is $\delta(t-t')$, by (80), which reduces the right hand side to an integral representation of a Bessel function [11, p. 48(4)]. Hence one obtains the identity

$$\sum_{n=0}^{\infty} \frac{|\nu_{\alpha n}(c)|^2}{\Lambda_{\alpha n}(c)} \psi_{\alpha n}(c, \eta)\psi_{\alpha n}(c, y) = 2^{\alpha+\frac{1}{2}}\, \Gamma(\alpha+1)\Gamma(\tfrac{1}{2}) \frac{J_{\alpha+\frac{1}{2}}(c[\eta-y])}{(c[\eta-y])^{\alpha+\frac{1}{2}}} \tag{82}$$

for $-\infty<\eta<\infty$, $-\infty<y<\infty$.

An important special case of (82) occurs when $y=\eta$, giving

$$\sum_{n=0}^{\infty} \frac{|\nu_{\alpha n}(c)|^2}{\Lambda_{\alpha n}(c)} \psi^2_{\alpha n}(c, \eta) = \frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2})}{\Gamma(\alpha+\frac{3}{2})} \tag{83}$$

for $-\infty<\eta<\infty$. At $\eta=0$ the product $\nu_{\alpha n}(c)\psi_{\alpha n}(c, 0)$ for odd n is zero and for even n is proportional to a_0, by (23), hence

$$\sum_{r=0}^{\infty} \frac{a_0^2(c|\alpha, 2r)}{\Lambda_{\alpha, 2r}(c)} = \tfrac{1}{2}\Gamma(2\alpha+2). \tag{84}$$

This identity provides a means for an overall numerical check on computed values of the normalization factors $\Lambda_{\alpha n}(c)$ alone for even n. To obtain the corresponding identity for odd n construct a new identity like (83),

$$\sum_{n=0}^{\infty} \frac{|\nu_{\alpha n}(c)|^2}{\Lambda_{\alpha n}(c)} \psi'^2_{\alpha n}(c, \eta) = c^2 \frac{\Gamma(\alpha+1)\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{5}{2})} \tag{85}$$

for $-\infty<\eta<\infty$. At $\eta=0$ this reduces, by (24), to

$$\sum_{r=0}^{\infty} \frac{a_1^2(c|\alpha, 2r+1)}{\Lambda_{\alpha, 2r+1}(c)} = (\alpha+\tfrac{3}{2})\Gamma(2\alpha+2), \tag{86}$$

which provides a means for checking numerical values of the $\Lambda_{\alpha n}(c)$'s alone for odd n.

An identity for the eigenvalues $\nu_{\alpha n}(c)$ alone can be obtained by multiplying (83) by $(1-\eta^2)^\alpha$ and integrating over $(-1, 1)$,

$$\sum_{n=0}^{\infty} |\nu_{\alpha n}(c)|^2 = \left(\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2})}{\Gamma(\alpha+\frac{3}{2})}\right)^2. \tag{87}$$

An identity for checking the new numbers $\gamma_{\alpha n}(c)$ for even n can be obtained from (82) by letting $y=0$, multiplying by $(\eta^2-1)^\alpha$, and integrating over $(1, \infty)$,

$$\sum_{r=0}^{\infty} (-1)^r \frac{|\nu_{\alpha, 2r}(c)|^3 \psi^2_{\alpha, 2r}(c, 0)}{\Lambda_{\alpha, 2r}(c)} [\gamma_{\alpha, 2r}(c)-1] = 2^{\alpha+\frac{3}{2}}\, \Gamma(\alpha+1)\Gamma(\tfrac{1}{2}) I_{\alpha 0}(c), \tag{88}$$

where $I_{\alpha 0}(c)$ is as defined by (52). For odd n differentiate (82) before letting $y=0$ and then multiply by $\eta(\eta^2-1)^\alpha$ and integrate over $(1, \infty)$,

$$\sum_{r=0}^{\infty} (-1)^{r+1} \frac{|\nu_{\alpha, 2r+1}(c)|^3 \psi'^2_{\alpha, 2r+1}(c, 0)}{\Lambda_{\alpha, 2r+1}(c)} [\gamma_{\alpha, 2r+1}(c)-1] = 2^{\alpha+\frac{3}{2}}\, \Gamma(\alpha+1)\Gamma(\tfrac{1}{2}) c^2 I'_{\alpha 0}(c). \tag{89}$$

Another identity can be constructed by multiplying integral equation (2) by $\psi_{\alpha n}(c, y)/\sqrt{\Lambda_{\alpha n}(c)}$ and summing,

$$\sum_{n=0}^{\infty} \frac{\nu_{\alpha n}(c)}{\Lambda_{\alpha n}(c)} \psi_{\alpha n}(c, \eta)\psi_{\alpha n}(c, y) = e^{ic\eta y}, \tag{90}$$

for $-\infty < \eta < \infty$, $-1 < y < 1$. For $y=0$ this becomes, from (23) and (24),

$$\sum_{r=0}^{\infty} (-1)^r \frac{a_0(c|\alpha, 2r)}{\Lambda_{\alpha, 2r}(c)} \psi_{\alpha, 2r}(c, \eta) = \frac{2^{\alpha}\Gamma(\alpha+\frac{3}{2})}{\Gamma(\frac{1}{2})}, \tag{91}$$

$$\sum_{r=0}^{\infty} (-1)^r \frac{a_1(c|\alpha, 2r+1)}{\Lambda_{\alpha, 2r+1}(c)} \psi_{\alpha,2r+1}(c, \eta) = \frac{2^{\alpha+1}\Gamma(\alpha+\frac{5}{2})}{\Gamma(\frac{1}{2})} \eta, \tag{92}$$

for $-\infty < \eta < \infty$.

In the limiting case where $c \to 0$ and η is such that $c\eta = \zeta$,

$$\psi_{\alpha n}(c, \eta) \to \frac{n!}{\Gamma(n+2\alpha+1)} a_n T_n^{\alpha}(\eta), \tag{93}$$

$$\Lambda_{\alpha n}(c) \to \frac{n!}{(n+\alpha+\frac{1}{2})\Gamma(n+2\alpha+1)} a_n^2, \tag{94}$$

$$|\nu_{\alpha n}(c)|\psi_{\alpha n}(c, \eta) \to \sqrt{2\pi}\, a_n \frac{J_{n+\alpha+\frac{1}{2}}(\zeta)}{\zeta^{\alpha+\frac{1}{2}}}. \tag{95}$$

Then (82) for $c \to 0$ with $c\eta = \zeta$ and $cy = z$ reduces to

$$\sum_{n=0}^{\infty} \frac{(n+\alpha+\frac{1}{2})\Gamma(n+2\alpha+1)}{n!} \frac{J_{n+\alpha+\frac{1}{2}}(\zeta)}{\zeta^{\alpha+\frac{1}{2}}} \frac{J_{n+\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}} = \frac{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)\Gamma(\frac{1}{2})}{2\pi} \frac{J_{\alpha+\frac{1}{2}}(\zeta-z)}{(\zeta-z)^{\alpha+\frac{1}{2}}}, \tag{96}$$

which is one of Gegenbauer's addition theorems [11, p. 363(2)], and (90) for $c \to 0$ with $c\eta = \zeta$ reduces to

$$\sqrt{2\pi} \sum_{n=0}^{\infty} i^n (n+\alpha+\tfrac{1}{2}) \frac{J_{n+\alpha+\frac{1}{2}}(\zeta)}{\zeta^{\alpha+\frac{1}{2}}} T_n^{\alpha}(y) = e^{i\zeta y}, \tag{97}$$

which is another [11, p. 368(2)].

9. A Special Case

For the special case in which $c = \frac{q\pi}{2}$, $\alpha = 1$, and $n = q-1$, with $q = 1, 2, 3, \ldots$, an exact closed-form representation can be obtained for the spheroidal function and each of its associated numbers. Its importance lies in the fact that it provides an independent means for an exact numerical check.

The spheroidal function sought can be obtained from the differential equation (1) and the condition of finiteness at $\eta = \pm 1$ by letting

$$\psi_{\alpha n}(c, \eta) = \frac{u_{\alpha n}(c, \eta)}{1-\eta^2}, \tag{98}$$

which leads to

$$(1-\eta^2)u''_{\alpha n}-2(\alpha-1)\eta u'_{\alpha n}+\left(b_{\alpha n}+2-c^2\eta^2-\frac{4(\alpha-1)\eta^2}{1-\eta^2}\right)u_{\alpha n}=0. \tag{99}$$

When $\alpha=1$ this reduces to

$$(1-\eta^2)u''_{1n}+(b_{1n}+2-c^2\eta^2)u_{1n}=0. \tag{100}$$

For $c=\frac{q\pi}{2}$ and assuming for the moment that

$$b_{1,\,q-1}\left(\frac{q\pi}{2}\right)=\left(\frac{q\pi}{2}\right)^2-2, \tag{101}$$

(100) becomes, for $n=q-1$,

$$u''_{1,\,q-1}+\left(\frac{q\pi}{2}\right)^2 u_{1,\,q-1}=0 \tag{102}$$

whose solution is

$$u_{1,\,q-1}\left(\frac{q\pi}{2},\,\eta\right)=k_q \begin{matrix}\cos\\ \sin\end{matrix}\,\frac{q\pi}{2}\,\eta, \tag{103}$$

hence

$$\psi_{1,\,q-1}\left(\frac{q\pi}{2},\,\eta\right)=k_q\,\frac{\begin{matrix}\cos\\ \sin\end{matrix}\,\frac{q\pi}{2}\,\eta}{1-\eta^2},\qquad q\ \begin{matrix}\text{odd}\\ \text{even}\end{matrix}, \tag{104}$$

where k_q is a normalizing constant. Expression (104) satisfies the differential equation (1) and is finite at $\eta=\pm 1$, so it is clearly a spheroidal function. Furthermore it has exactly $q-1$ zeros in the interval $(-1,\,1)$, so the index of the spheroidal function must be $q-1$. Thus the validity of (101) is established. The spheroidal function (104) and its eigenvalue (101) were obtained earlier by Flammer [17].

The spheroidal function (104) satisfies also the integral equation (2), which can be verified by direct calculation of the integral,

$$\int_{-1}^{1} e^{i\frac{q\pi}{2}\eta t}\,(1-t^2)\,\frac{\begin{matrix}\cos\\ \sin\end{matrix}\,\frac{q\pi}{2}\,t}{1-t^2}\,dt=i^{q-1}\,\frac{4}{q\pi}\,\frac{\begin{matrix}\cos\\ \sin\end{matrix}\,\frac{q\pi}{2}\,\eta}{1-\eta^2}. \tag{105}$$

This is the integral equation in question, from which it is evident that its eigenvalue is

$$\nu_{1,\,q-1}\left(\frac{q\pi}{2}\right)=i^{q-1}\frac{4}{q\pi}. \tag{106}$$

The normalization factor on $(-1,\,1)$ in terms of the constant k_q is found by direct calculation of the integral (42) to be

$$\Lambda_{1,\,q-1}\left(\frac{q\pi}{2}\right)=\frac{k_q^2}{2}\,\mathrm{Cin}\,2q\pi. \tag{107}$$

And by direct calculation of the integral (46) one finds

$$\gamma_{1,q-1}\left(\frac{q\pi}{2}\right)=2. \tag{108}$$

The above results provide an interesting indication of the way in which the integral in the definition (45) of $\gamma_{\alpha n}(c)$ can behave outside of the open interval $-1<t<1$. For q odd, for example, the left-hand side of (45) can be evaluated in closed form,

$$(-1)^{\frac{q-1}{2}}\frac{q\pi}{4}\int_{-\infty}^{\infty}e^{i\frac{q\pi}{2}\eta t}|1-\eta^2|\frac{\cos\frac{q\pi}{2}\eta}{1-\eta^2}d\eta$$

$$=2\frac{\cos\frac{q\pi}{2}t}{1-t^2}-(-1)^{\frac{q-1}{2}}\frac{\pi}{2}[\delta(t-1)+\delta(t+1)], \tag{109}$$

which agrees exactly with the right-hand side of (45) within the open interval $-1<t<1$. It does not agree at the endpoints $t=\pm 1$, however, as evidenced by the delta function singularities that appear there.

10. Numerical Results

The formulas developed above have been used to compute and check the spheroidal functions and their associated numbers for values of c up to 5π. Representative numerical results for $c=6$ are shown below, all of which were computed on the Triangle Universities IBM 360/75 computer by Guy A. Myers at the North Carolina State University.

The first nine of the spheroidal functions $\psi_{\alpha n}(c,\eta)$ are shown in figure 1 for $\alpha=-\frac{1}{2}, 0, \frac{1}{2}, 1, 2,$ and 3. They were normalized on the interval $(-\infty, \infty)$ by choosing $\Lambda_{\alpha n}(c)$ to be $1/\gamma_{\alpha n}(c)$. The corresponding functions for $\alpha=0$ with $c=0.5$, 1, 2, and 4 were shown earlier by Slepian and Pollak [12]. The characteristic numbers $b_{\alpha n}(c)$, $\nu_{\alpha n}(c)$, and $\gamma_{\alpha n}(c)$ as a function of α are shown in figures 2, 3, and 4, respectively.

Numerical results for a few representative cases are shown in tables 1 and 2. Table 1 shows computed values of $b_{\alpha n}(c)$, $|\nu_{\alpha n}(c)|$, and $\psi_{\alpha n}(c,0)$ for even n and $\psi'_{\alpha n}(c,0)$ for odd n, for $\alpha=-\frac{1}{2}$, 0, and $+\frac{3}{4}$ at $c=6$, $n=0$ to 13. Table 2 shows computed values for the reciprocal of $\gamma_{\alpha n}(c)$ for $\alpha=-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, and $\frac{5}{4}$, at $c=6$, $n=0$ to 13. The reciprocal of $\gamma_{\alpha n}(c)$, rather than $\gamma_{\alpha n}(c)$ itself, is shown in order to illustrate the relative concentration (79) of the functions.

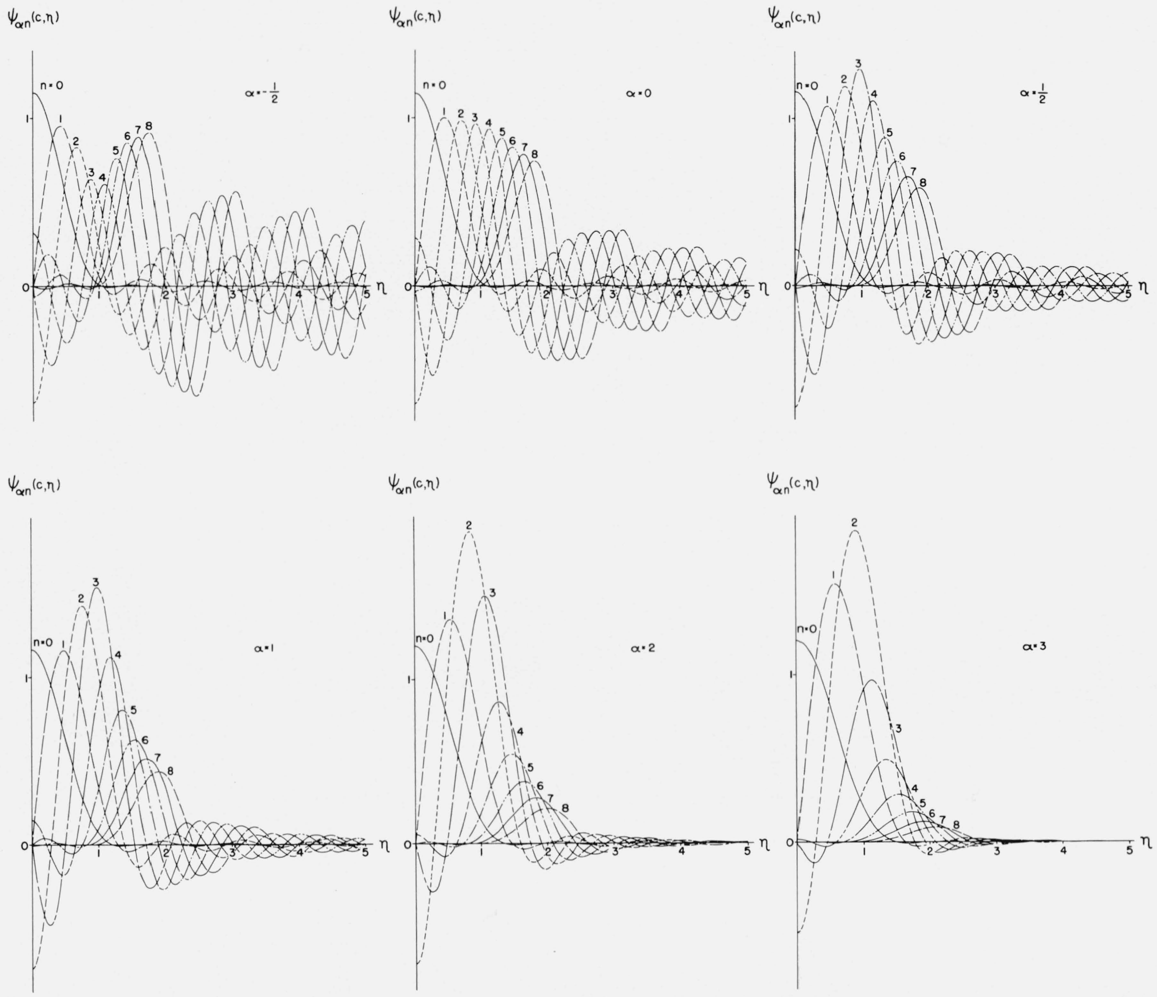

FIGURE 1. *The spheroidal functions for* $\alpha = -1/2, 0, 1/2, 1, 2, 3$, $c=6$, $n=0$ to 8, *with* $\Lambda_{\alpha n}(c) = 1/\gamma_{\alpha n}(c)$.

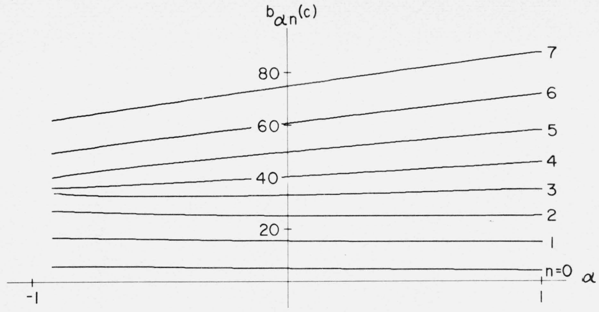

FIGURE 2. *The eigenvalues* $b_{\alpha n}(c)$ *of the differential equation* (1) *for* $-0.92 \leq \alpha \leq 1$, $c=6$.

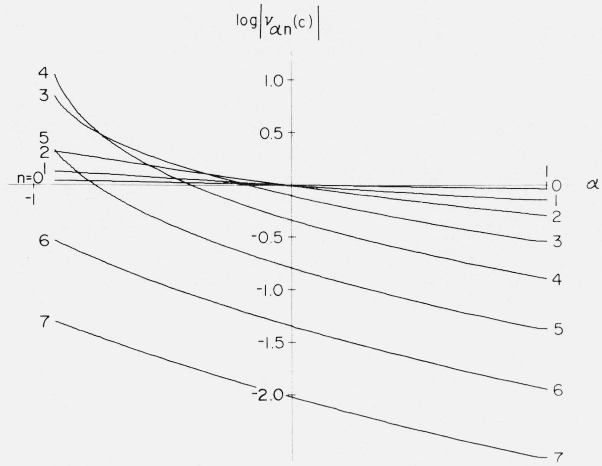

FIGURE 3. *The eigenvalues* $\nu_{\alpha n}(c)$ *of the integral equation* (2) *for* $-0.92 \leq \alpha \leq 1$, $c=6$.

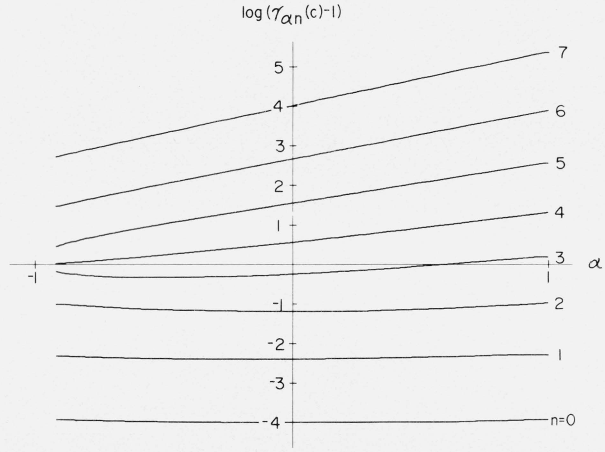

FIGURE 4. *The new numbers* $\gamma_{\alpha n}(c)$ *defined by* (45) *for* $-0.92 \leq \alpha \leq 1$, $c=6$.

TABLE 1

n	$b_{\alpha n}(c)$	$\lvert\nu_{\alpha n}(c)\rvert$	$\psi_{\alpha n}(c, 0)$ $\psi'_{\alpha n}(c, 0)$
		$\alpha=-1/2, \quad c=6$	
0	0.5737585781+01	0.1074447786+01	0.1155255760+01
1	.1661329843+02	.1212324184+01	.3686243582+01
2	.2598284316+02	.1469587054+01	−.6927129388+00
3	.3290367966+02	.1770806514+01	−.2666895609+01
4	.3816092638+02	.1311549557+01	.3176164384+00
5	.4509186607+02	.4656092304+00	.1273298762+01
6	.5521934384+02	.1117504031+00	−.6410649522−01
7	.6785668990+02	.2310300898−01	−.1308328538+00
8	.8264753135+02	.4243576673−02	.3284620306−02
9	.9950842599+02	.6981239327−03	.5920679880−02
10	.1184102166+03	.1037825612−03	−.9512187852−04
11	.1393381242+03	.1405973769−04	−.1623325826−03
12	.1622835823+03	.1748761994−05	.1808459022−05
13	.1872412953+03	.2010009631−06	.2990162450−05
		$\alpha=0, \quad c=6$	
0	0.5208269159+01	0.1023276503+01	0.1153430212+01
1	.1600044275+02	.1021309607+01	.3662488818+01
2	.2535647864+02	.9922435547+00	−.7045705538+00
3	.3320419949+02	.8229938914+00	−.2977793219+01
4	.4072019426+02	.4659781026+00	.2864309440+00
5	.4977371212+02	.1693510360+00	.7958373899+00
6	.6118075689+02	.4524678973−01	−.3152431639−01
7	.7485286652+02	.9966212672−02	−.6222320400−01
8	.9065115936+02	.1897100693−02	.1388242306−02
9	.1085154453+03	.3192404741−03	.2472548565−02
10	.1284188799+03	.4820488557−04	−.3606307265−04
11	.1503474434+03	.6605196410−05	−.6113574747−04
12	.1742930029+03	.8286721621−06	.6275570287−06
13	.2002505133+03	.9588946460−07	.1033409530−05
		$\alpha=+3/4, \quad c=6$	
0	0.4524868470+01	0.9514693801+00	0.1157683567+01
1	.1554416968+02	.7960998515+00	.3741186029+01
2	.2556039130+02	.6000021716+00	−.7319049126+00
3	.3502025958+02	.3687113331+00	−.2915597899+01
4	.4493678220+02	.1690739803+00	.1800330864+00
5	.5631678092+02	.5762489345−01	.3736554493+00
6	.6963816767+02	.1546055800−01	−.1125847285−01
7	.8500027646+02	.3450023959−02	−.2071854851−01
8	.1024006606+03	.6633536963−03	.3879838010−03
9	.1218275965+03	.1124131264−03	.6700408720−03
10	.1432726639+03	.1705749626−04	−.8491880927−05
11	.1667303958+03	.2345366168−05	−.1414240391−04
12	.1921972059+03	.2949700757−06	.1289490027−06
13	.2196706817+03	.3419275479−07	.2099163320−06

TABLE 2

Reciprocal of $\gamma_{\alpha n}(c)$ for $c=6$

n	$\alpha=-3/4$	$\alpha=-1/2$	$\alpha=-1/4$
0	0.9998903116 + 00	0.9998968961 + 00	0.9999006633 + 00
1	.9953591572 + 00	.9957735784 + 00	.9959984111 + 00
2	.9251077487 + 00	.9361280337 + 00	.9403541758 + 00
3	.6617807432 + 00	.6867880908 + 00	.6773573623 + 00
4	.4329978267 + 00	.3559391049 + 00	.2783892137 + 00
5	.1631458857 + 00	.8787799051 − 01	.4890053549 − 01
6	.1812019930 − 01	.8375304321 − 02	.4009596286 − 02
7	.1085823478 − 02	.4742740153 − 03	.2109141235 − 03
8	.4704479809 − 04	.1948721509 − 04	.8155744837 − 05
9	.1566525909 − 05	.6167222470 − 06	.2444286528 − 06
10	.4138876101 − 07	.1554061438 − 07	.5862729444 − 08
11	.8891403564 − 09	.3195449002 − 09	.1152410426 − 09
12	.1584025115 − 10	.5466463632 − 11	.1891529381 − 11
13	.2378498905 − 12	.7904581458 − 13	.2632527573 − 13

n	$\alpha=0$	$\alpha=+1/4$	$\alpha=+1/2$
0	0.9999018826 + 00	0.9999006439 + 00	0.9998968665 + 00
1	.9960616432 + 00	.9959753979 + 00	.9957367356 + 00
2	.9401733901 + 00	.9363969465 + 00	.9290353072 + 00
3	.6467919491 + 00	.5993323516 + 00	.5374082010 + 00
4	.2073492168 + 00	.1474458621 + 00	.1005292486 + 00
5	.2738716624 − 01	.1530259540 − 01	.8498117614 − 02
6	.1955000733 − 02	.9617772918 − 03	.4747110329 − 03
7	.9484876556 − 04	.4292801980 − 04	.1948802000 − 04
8	.3436783286 − 05	.1454316917 − 05	.6167231119 − 06
9	.9732115988 − 07	.3886092897 − 07	.1554061496 − 07
10	.2218980545 − 08	.8416064934 − 09	.3195449004 − 09
11	.4166226283 − 10	.1508525403 − 10	.5466463632 − 11
12	.6557478591 − 12	.2276034041 − 12	.7904581458 − 13
13	.8780377122 − 14	.2931302844 − 14	.9790506399 − 15

n	$\alpha=+3/4$	$\alpha=+1$	$\alpha=+5/4$
0	0.9998902878 + 00	0.9998804332 + 00	0.9998665602 + 00
1	.9953272353 + 00	.9947106203 + 00	.9938277694 + 00
2	.9175535778 + 00	.9009322432 + 00	.8776555430 + 00
3	.4640663909 + 00	.3839718982 + 00	.3032540474 + 00
4	.6605749034 − 01	.4205022091 − 01	.2605099768 − 01
5	.4681647613 − 02	.2555876781 − 02	.1381915965 − 02
6	.2342324766 − 03	.1152638318 − 03	.5647610537 − 04
7	.8852297954 − 05	.4016362284 − 05	.1817713005 − 05
8	.2616675768 − 06	.1109412441 − 06	.4695584821 − 07
9	.6217024284 − 08	.2485740919 − 08	.9925660313 − 09
10	.1213549245 − 09	.4606583810 − 10	.1746778699 − 10
11	.1981169523 − 11	.7177204946 − 12	.2597757239 − 12
12	.2745434078 − 13	.9531862327 − 14	.3306789087 − 14
13	.3270100000 − 15	.1091853480 − 15	.3643090846 − 16

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