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Selecting Nonlinear Transformations for the Evaluation of Improper integrals*

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Recent literature concerning the use of nonlinear transformations to evaluate numerically certain improper integrals of the first kind involves the determination of a transformation function g to improve the approximation. By approximating a given integrand f by an integrable function f_1 and then determining an associated g function for f_1, a nonlinear transformation may be constructed which will yield an improved approximation of the improper integral of f.

Key words: Improper integral, nonlinear transformation.

1. Introduction

H. L. Gray and T. A. Atchison[1] introduced a class of nonlinear transformations to assist in the numerical evaluation of improper integrals of the first kind. A transformation from this class is completely determined by specifying the integrand of the improper integral and a transformation function g satisfying certain mild restrictions. The purpose of this paper is to exhibit a scheme for the selection of g which will yield a good approximation to the improper integral.

2. Selecting the Transformation

The Generalized G-transform of Atchison and Gray is as follows:
If f is continuous on $[a, \infty)$ and

$$F(t)=\int_a^t f(x)dx \to S \text{ as } t \to \infty, \tag{2.1}$$

then

$$G[F; g; t]=\frac{F(t)-R(t)F(g(t))}{1-R(t)} \tag{2.2}$$

where $g \epsilon C^1$ and $g(t) \geqslant a$ on $[a, \infty)$, $\lim_{t\to\infty} g(t)=\infty$, and

$$R(t)=\frac{f(t)}{f(g(t))g'(t)} \tag{2.3}$$

One of the basic theorems proved (see footnote 1) concerns the determination of a differentiable function g such that $G[F; g; t] \equiv S$ for all $t \geqslant t_0 \geqslant a$ for some class of functions f:
THEOREM: *A necessary and sufficient condition that* $G[F; g; t] \equiv S$ *for all* $t \geqslant t_0 \geqslant a$ *is that* $R(t)$ *is constant for* $t \geqslant t_0$.

In general, for a given integrand f, the determination of a function g for which $R(t)$ is constant is difficult. However, suppose f_1 is an integrable function which approximates f. Requiring the

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[1] H. L. Gray and T. A. Atchison, The Generalized G-transformation, Math. Comp. **22**, No. 103, 595–606 (1968).

corresponding $R(t)$ to be constant we get

$$f_1(g(t))g'(t)=c_1 f_1(t) \qquad \text{for } t \geqslant t_0. \tag{2.4}$$

If $y=g(t)$, then (2.4) becomes

$$f_1(y)\,dy=c_1 f_1(t)\,dt \tag{2.5}$$

and integration yields

$$\int f_1(y)\,dy=c_1\int f_1(t)\,dt+c_2, \tag{2.6}$$

where c_1 and c_2 are constants. Thus, if

$$H(y)=\int f_1(y)\,dy, \tag{2.7}$$

then any differentiable function g which satisfies the functional equation

$$H(y)=c_1 H(t)+c_2 \qquad \text{for } t \geqslant t_0. \tag{2.8}$$

will result in

$$G[F_1;\ g;\ t] \equiv S_1 \qquad \text{for } t \geqslant t_0. \tag{2.9}$$

where

$$F_1(t)=\int_a^t f_1(x)\,dx \to S_1 \text{ as } t \to \infty. \tag{2.10}$$

Since f_1 approximates f, then $G[F_1;\ g;\ t]$ will be an approximation of S.

To illustrate this procedure, consider

$$F(t)=\frac{2}{\sqrt{\pi}}\int_a^t e^{-x^2}\,dx \to \operatorname{erfc}(a) \qquad \text{as } t \to \infty, \tag{2.11}$$

where $0 < a < \infty$. For $y \geqslant 1$, the integrand of (2.11) may be approximated by

$$f_1(y)=ye^{-y^2} \tag{2.12}$$

and f_1 possesses an integral

$$H(y)=-\frac{1}{2}e^{-y^2}. \tag{2.13}$$

One may determine a function $y=g(t)$ which satisfies (2.8) by choosing $c_1=e^{-k^2}$ and $c_2=0$. Such a function is

$$g(t)=\sqrt{t^2+k^2}. \tag{2.14}$$

Applying (2.2) to (2.11) and utilizing (2.14) we get

$$G[F; g; t] = \frac{F(t) - e^{k^2}\dfrac{\sqrt{t^2+k^2}}{t}F(\sqrt{t^2+k^2})}{1 - e^{k^2}\dfrac{\sqrt{t^2+k^2}}{t}} \tag{2.15}$$

Theorem 1 (see footnote 1) may be applied to (2.15) to show that it converges to erfc (a) more rapidly than either $F(t)$ on $F(\sqrt{t^2+k^2})$. This is illustrated by the following numerical values in the case $a=1$, $k=1$:

t	$F(t)$	Error	$G[F; g; t]$	Error
1.0	0.0	0.15729921	0.15023133	0.00706788
1.5	.12340436	.03389485	.15655151	.00074770
2.0	.15262147	.0046777	.15745495	.00015574

If we apply Aitken's Δ^2-process to the $F(t)$ column [2] the resulting approximation is 0.16168462 which is in error by 0.00438541.

A general procedure for solving the functional equation (2.8) is not available. Also, for different choices of c_1 and c_2, other solutions of the functional equation may be determined.

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[2] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, NBS Applied Mathematics Series 55, U.S. Government Printing Office.