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Normal Subgroups of the Modular Group*

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A number of results on the normal subgroup structure of the classical modular group is announced. A typical result is that a normal subgroup of square-free index is necessarily of genus 1, apart from 4 exceptions.

Key words: Genus; index; modular group; normal subgroups.

1. Introduction

In this note we summarize the results of some work on the normal subgroups of the classical modular group Γ, which is a continuation of the work begun in [1] [1] and [4]. We may regard Γ as the free product of a cyclic group of order 2 and a cyclic group of order 3; $\Gamma = \{x\} * \{y\}, x^2 = y^3 = 1$. The number of normal subgroups of Γ of index μ will be denoted by $N(\mu)$. If G is any subgroup of Γ, G' will denote its commutator subgroup, and G^p the fully invariant subgroup of G generated by the pth powers of the elements of G. The *level* of G is the least positive integer n such that $(xy)^n \epsilon G$. If G is a normal subgroup of index $\mu \geq 6$ and n is its level, then the genus of G is given by

$$g = 1 + \mu(n-6)/12n,$$

and the number of parabolic classes of G by

$$t = \mu/n.$$

Except for the groups Γ, Γ^2, or Γ^3, the index of a normal subgroup is a multiple of 6.

The commutator subgroup Γ' of Γ is a free group of rank 2, freely generated by

$$a = xyxy^2, \qquad b = xy^2xy.$$

The normal subgroups of Γ of genus 1 (alternatively, of level 6) have been completely described in [5]. Any such subgroup G lies between Γ' and Γ'' and may be described uniquely by the triplet of integers (p, m, d), where $p > 0$, $0 \leq m \leq d-1$, $m^2 + m + 1 \equiv 0 \bmod d$. G is of index $6dp^2$ in Γ and consists of all words w of Γ' satisfying

$$e_a(w) \equiv 0 \bmod p, \qquad e_b(w) \equiv me_a(w) \bmod dp,$$

where $e_a(w)$, $e_b(w)$ are the respective exponent sums in a and b of w.

We also let $G_{k,m}$ be the intersection of all normal subgroups of Γ containing

$$(xy)^{mk}, \qquad (yx)^k(xy)^{-k}.$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

Going over to the representation of Γ as $LF(2, Z)$, we define the *principal congruence subgroup* $\Gamma(n)$ as the totality of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \Gamma$ such that

$$a \equiv d \equiv \pm 1 \bmod n, \qquad b \equiv c \equiv 0 \bmod n.$$

2. The Results

We now state the principal results obtained. Throughout this section G is a normal subgroup of Γ of index μ, level n, genus g, and having t parabolic classes.

(1) Suppose that μ is square-free. Then either $G=\Gamma$, Γ^2, Γ^3, or $\Gamma(2)$, or else G is of genus 1 and every prime divisor of $\mu/6$ is $\equiv 1 \bmod 3$.

(2) Define $f(\mu)$ as 1 if there is a normal subgroup G of index μ with solvable quotient group Γ/G, and 0 otherwise. Then

$$\lim_{x \to \infty} \frac{1}{x} \sum_{\mu \leq x} f(\mu) = 0.$$

(3) Let p be a prime, $p \equiv -1 \bmod 3$, and suppose that $p > r$. Then there is no normal subgroup of Γ of index pr.

(4) If Γ/G is nilpotent then it is abelian, and G must be Γ, Γ^2, Γ^3, or Γ'.

(5) Let p be a prime, $p \equiv 1 \bmod 12$. Then there are no normal subgroups of Γ having $2p$ parabolic classes.

(6) Let p be a prime > 84, $p \equiv -1 \bmod 3$; and let n be any positive integer. Then there are no normal subgroups of Γ of genus $1+p^n$.

(7) Let p be a prime >5, and suppose that $\mu=6p^2$. Then G must be one of the following groups: groups:

(i) $\Gamma(2)^p\Gamma(2)'$.

(ii) $(p, 0, 1)$.

(iii) $(1, m_1, p^2)$, $(1, m_2, p^2)$, where $p \equiv 1 \bmod 3$ and m_1, m_2 are the solutions of $m^2+m+1 \equiv 0 \bmod p^2$.

Thus

$$N(6p^2) = 3 + (p/3), \quad p \text{ prime}, p > 5.$$

(8) Let p be a prime > 11. Then $N(12p^2) = 0$.

(9) Let p be a prime > 11. Then the only normal subgroup of Γ of index $12p^3$ is $\Gamma(3)^p\Gamma(3)'$.

(10) $N(72)=2$, $N(78)=2$, $N(84)=0$, $N(90)=0$.

(11) There is just one normal subgroup of Γ of genus 2: namely $G_{4,2}$.

(12) The normal subgroups of Γ with t parabolic clasese, $t \leq 5$, are the following:

$t=1$: Γ, Γ^2, Γ^3, Γ'.

$t=2$: none.

$t=3$: $\Gamma(2)$, $(1, 1, 3)$.

$t=4$: $(2, 0, 1)$, $\Gamma(3)$, $G_{3,4}$.

$t=5$: none.

3. Some Remarks

Perhaps the most striking results are the first two. A generalization of (2) with a precise estimate for the density function is in course of publication ([2]). As for (1), we note that if G is any finite group of square-free order generated by elements x, y such that $x^2=y^3=1$, then $(xy)^6=1$. This is so since the second commutator subgroup G'' is necessarily $\{1\}$ (p. 148 of [3]), and

$$a = xyxy^2 \epsilon G', \quad b = xy^2xy \epsilon G',$$

$$(xy)^6 = ab^{-1}a^{-1}b \epsilon G''.$$

The result (1) is now an easy consequence.

The other results are of varying degrees of difficulty, but generally present no special problems.

4. References

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